University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

Tuesday, May 26, 2015

9:00 AM – 3:00 PM

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth two marks each. Answer all questions in Part A on the question paper itself. Each of these questions can and should be answered in no more than five sentences.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 8 questions out of 12 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark respectively the first 8 questions for Part B, and first 4 questions for Part C in the order of their appearance in your answer booklets.
- In order to pass this examination, you must obtain a score of at least 70% in each of the three parts; that is to say, $14/20$ in Part A, $28/40$ in Part B, and $28/40$ in Part C.

PART A

Please answer each of the following 10 questions in the space provided. Each correct answer is worth two marks. Each question should be answered briefly; i.e., in no more than five sentences.

Q1. Show that the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is not diagonalizable.

Q2. Let K denote a finite field. Show that K is not algebraically closed.

Q3. Let G be a group. Define $\phi : G \to G$ such that $\phi(g) = g^{-1}$. Prove that if ϕ is a homomorphism then G is abelian.

Q4. Prove that the polynomial

$$x^3 + 3\,x^2 + 8\,x + 21$$

is irreducible over $\mathbb{Q}$. (Hint: Let $x = y - 1$)

Q5. Let p be a prime number and $\mathbb{F}_p$ a finite field with p elements. Prove or disprove: The polynomial $x^p + a$ is irreducible over $\mathbb{F}_p$ for any nonzero $a \in \mathbb{F}_p$.

Q6. Let V be a real vector space and $L : V \to \mathbb{R}$ a nonzero linear map. Suppose that $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\ker L$. Given any nonzero vector $v \notin \ker L$, show that $\{v, v_1, \ldots, v_n\}$ is a basis for V.

Q7. Suppose that H and K are finite subgroups of a group G. Show that $\gcd(|H|, |K|) = 1$ implies $H \cap K = \{e\}$.

Q8. Let S_4 denote the group of permutations on the set $\{1, 2, 3, 4\}$. What is the subgroup generated by $(1, 2, 3)$ and $(1, 2)$? Why?

Q9. Let R be a ring. Give a careful definition of a flat R-module.

Q10. Prove or disprove: If R is a nonzero subring of $\mathbb{Z}_{15}$, then $R = \mathbb{Z}_{15}$.

PART B

Please answer any 8 of the following 12 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 8, then please indicate clearly which ones you want us to mark.

Q1. Prove that we have an isomorphism of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_9\,,\,\mathbb{Z}_{12}) \simeq \mathbb{Z}_3.$$

Q2. Let x_1, x_2 denote independent variables, and consider the elementary symmetric polynomials

$$s_1 = x_1 + x_2, \qquad s_2 = x_1\,x_2.$$

Express $x_1^5 + x_2^5$ as a polynomial in s_1 and s_2. (This is an instance of the 'Fundamental Theorem on Symmetric Polynomials'.)

Q3. Let G be a simple group of order 168. (You may assume without proof that such a group exists.)

(a) Prove that the number of 7-Sylow subgroups of G is 8.

(b) Determine the number of elements of order 7 in G.

Q4. Show that any finite integral domain is a field.

Q5. Let p be a prime. Prove that a group of order p^n has non trivial centre.

Q6. Let $\alpha = \sqrt{3} + i$, where $i = \sqrt{-1}$.

- Prove that we have an equality of fields

$$\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{3}, i).$$

- Give a careful definition of a finite Galois extension, and prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension.

Q7. Let z denote a real number.

- Give a careful definition of what it means to say that z is a constructible number over $\mathbb{Q}$.

- Prove that $\cos \frac{\pi}{9}$ is not constructible. Hint: Use the formula

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Q8. Let $R = \mathbb{Q}[x, y]$ and consider the ideal $I = \{f \in R : \ f(0,0) = f(1,1) = 0\}$. Prove that

$$I = \langle\, x - y \,,\, y - x^2 \,\rangle.$$

Q9. Let V be a finite-dimensional $\mathbb{R}$-vector space, and $T : V \longrightarrow V$ a linear transformation such that

$$\text{range}(T) = \text{nullspace}(T).$$

Prove that $\dim(V)$ is even. Give an example of such a T and V, where $\dim(V) = 4$.

Q10. Give a careful definition of an Artin commutative ring. Prove that the ring

$$\frac{\mathbb{Q}[x, y, z]}{(x^2, y^3, z^7)}$$

is Artin.

Q11. Give a careful definition of a solvable group, and show that every subgroup of a solvable group is solvable.

Q12. Consider the following basis of $\mathbb{R}^3$:

$$B = \{(1, 1, 0), (1, 2, 0), (0, 1, 2)\}.$$

Apply the Gram-Schmidt process to find an orthonormal basis of $\mathbb{R}^3$.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4, then please indicate clearly which ones you want us to mark.

Q1. Let $\mathbb{F}_3$ denote the field $\{0, 1, 2\}$, and consider the polynomial $p(t) = t^3 + 2\,t + 1 \in \mathbb{F}_3[t]$.

- Prove that $p(t)$ is irreducible over $\mathbb{F}_3$.
- Let
$$K = \frac{\mathbb{F}_3[t]}{(p(t))}$$
denote the corresponding field, seen as an $\mathbb{F}_3$-vector space. Consider the morphism
$$K \xrightarrow{f} K, \quad f(z) = z^3.$$
Prove that f is a linear transformation.
- Find the matrix of f with respect to the basis $\{1, t, t^2\}$ of K.

Q2. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers and let $\alpha = 5 - i$.

- Prove that we have an isomorphism of rings $\dfrac{R}{\langle \alpha \rangle} \simeq \mathbb{Z}_{26}$.
- Find a factorization of α into Gaussian primes.

Q3. Let R be a commutative ring, and let M, N be R-modules.

- Give a precise and careful definition of the tensor product
$$M \otimes_R N$$
and describe the R-module structure on it.
- Let I, J be ideals in R such that $I + J = (1)$. Prove that there is an isomorphism of R-modules
$$\frac{R}{I} \otimes_R \frac{R}{J} \simeq (0).$$

Q4.The Heisenberg group over a field F is the multiplicative group of matrices

$$H(F) = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \,\middle|\, a, b, c \in F \right\}.$$

- Show that $H(F)$ is never abelian for any choice of F.
- Determine which matrices are in the centre $Z(H(F))$ and show that $Z(H(F))$ is isomorphic to the additive group underlying the field F.
- Show that when F is the field with two elements, $H(F)$ is isomorphic to the dihedral group D_4.

Q5. Let P_3 denote the vector space of polynomials in x of degree less than or equal to 3, with real coefficients. Consider the linear map

$$P_3 \xrightarrow{L} P_3, \qquad L(f) = x\, f'' + 2\, f.$$

- Find the matrix M of L with respect to some basis of P_3.
- Find the Jordan canonical form of M, and corresponding basis of P_3.

Q6. Let R be a ring, and let

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of left R-modules. Given a left R-module T, prove that the induced sequence

$$0 \longrightarrow \mathrm{Hom}(T, M) \xrightarrow{\tilde{f}} \mathrm{Hom}(T, N) \xrightarrow{\tilde{g}} \mathrm{Hom}(T, P)$$

is exact. Give an example to show that, in general, $\tilde{g}$ is not surjective.

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