University of Manitoba
Department of Mathematics

Graduate Comprehensive Examination in Algebra

May 1, 2017 (9 am - 3 pm)

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Instructions (Please read carefully):

- You have altogether 6 hours to complete the examination.
- Part A consists of 10 questions worth 2 mark each. Answer all questions in Part A on the question paper itself.
- You have a choice of questions in each of Parts B and C. The questions in Part B are worth 5 marks each. Answer any 8 questions out of 12 in this part. The questions in Part C are worth 10 marks each. Answer any 4 questions out of 6 in this part.
- You may attempt as many questions as you like in Parts B and C; however, if you attempt more than the required number of questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark respectively the first 8 questions for Part B, and first 4 questions for Part C in the order of their appearance in your answer booklets.
- In order to pass this examination, you must obtain a score of at least 75% in total.

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I have submitted _______ numbered pages together with this question paper.

Student's Signature:

Invigilator's Signature:

PART A

Please answer each of the following 10 questions briefly in the space provided. Each correct answer is worth 2 marks. No lengthy explanations are required.

Q1. Let $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$. What is the minimal polynomial of A?

Q2. Let R be a ring. Define what it means to say that M is a projective left R-module.

Q3. Give an example of a vector space V and a **non-zero** linear operator $T : V \longrightarrow V$, such that $\ker(T)$ is isomorphic to V.

Q4. Give an example of an infinite field of positive characteristic.

Q5. Define precisely what it means to say that a field extension $F \subseteq K$ is normal.

Q6. Let G be a group such that no two distinct elements $x, y \in G$ are conjugate to each other. Prove that G must be abelian.

Q7. Give a precise definition of a 'nilpotent group'.

Q8. Consider the vectors $v_1 = (1, -2, 1), v_2 = (1, 0, 1)$ in $\mathbb{R}^3$ with the usual inner product, and let $V = \text{Span}\,\{v_1, v_2\}$. Find an orthonormal basis of V by applying the Gram-Schmidt process to $\{v_1, v_2\}$.

Q9. Consider the quadratic form

$$Q(x, y) = x^2 + xy - 2y^2$$

over $\mathbb{R}$. Determine whether Q is positive definitive, negative definitive or indefinite.

Q10. Let F be a field with 125 elements. It is known (and you may assume) that the multiplicative group F^* is cyclic. How many cyclic generators does F^* have?

PART B

Please answer any 8 of the following 12 questions in your answer booklet. Each question is worth 5 marks. If you attempt more than 8, then please indicate clearly which ones you want us to mark.

Q1. Consider the abelian group

$$G = \mathbb{Z}_{81} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3,$$

of order 3^7. Let α_n denote the number of elements of order n in G. Find α_3 and α_9.

Q2.

- Define precisely what it means to say that a commutative ring R is a Euclidean domain.
- Now prove that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean domain.

Q3. Prove that a group of order 15 must be abelian.

Q4. Prove that we have an isomorphism of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_6, \mathbb{Z}_4) \simeq \mathbb{Z}_2.$$

Q5. If $\theta = e^{i\,\pi/4}$, prove that $\mathbb{Q} \subseteq \mathbb{Q}(\theta)$ is a Galois extension. Find the Galois group explicitly, and describe its action on θ.

Q6. Let A be a real symmetric matrix. Prove that there exists a real symmetric matrix B such that $B^3 = A$.

Q7. Consider the ring

$$A = \frac{\mathbb{Q}[x,y]}{(x^2+y^2)}.$$

- Prove that A is an integral domain.
- Now let F be the quotient field of A. Find a transcendence basis of F over $\mathbb{Q}$. What is the transcendence degree of F over $\mathbb{Q}$?

Q8. Find the number of irreducible polynomials of degree 5 over the field $\mathbb{F}_2 = \{0, 1\}$. You may use basic facts from the theory of finite fields, but you should state them clearly.

Q9. Let A be a 2×2 unitary matrix with entries in the field of complex numbers. Prove that

$$A = \begin{bmatrix} z & w \\ -\overline{w}\, e^{i\theta} & \overline{z}\, e^{i\theta} \end{bmatrix},$$

where z, w are some complex numbers such that $|z|^2 + |w|^2 = 1$, and θ is a real number.

Q10. Let t_1, t_2, t_3 be variables, and define

$$u = t_1 + t_2 + t_3, \quad v = t_1\, t_2 + t_2\, t_3 + t_3\, t_1, \quad w = t_1\, t_2\, t_3.$$

Express $t_1^3 + t_2^3 + t_3^3$ as a polynomial in u, v, w. (This is an instance of the 'fundamental theorem on symmetric functions'.)

Q11. Give an example *each* of a commutative ring R, and a nonzero left R-module M such that

(a) M is a flat R-module.

(b) M is a not a flat R-module.

You should explain in brief why your examples work.

Q12.

- Define precisely what it means to say that M is an injective left R-module over a ring R.
- Give an example of a module M over $R = \mathbb{Z}$, which is not injective. You should explain why your example works.

PART C

Please answer any 4 of the following 6 questions in your answer booklet. Each question is worth 10 marks. If you attempt more than 4, then please indicate clearly which ones you want us to mark.

Q1. Let F denote the finite field $\{0, 1, 2\}$, and consider the polynomial

$$p(x) = x^3 + 2\,x^2 + 2\,x + 2 \in F[x].$$

(a) Prove that $p(x)$ is irreducible over F, and hence

$$K = \frac{F[x]}{(p(x))}$$

is a field with 27 elements.

(b) What is the Galois group $G = \mathrm{Gal}(K/F)$? Describe the action of its elements on K.

(c) Find the minimal polynomial of $x + 1 \in K$ over F.

Q2.

(a) Give a precise definition of a 'solvable group'.

(b) Prove that a group of order 225 must be solvable.

(c) Which is the non-solvable group of smallest order? (You may simply quote the example; no proof is required.)

Q3. Let R be a ring. Let M be a right R-module, and N a left R-module.

(a) Give a careful construction of the tensor product $M \otimes_R N$.

(b) Consider $\mathbb{Z}_3$ and $\mathbb{Z}_5$ as $\mathbb{Z}$-modules in the usual way. Prove that $\mathbb{Z}_3 \otimes_{\mathbb{Z}} \mathbb{Z}_5$ is the zero module.

Q4.

(a) Let α be a real irrational number. Define carefully what it means to say that α is a constructible number over $\mathbb{Q}$.

(b) Prove that if α and β are constructible, then so is $\alpha + \beta$.

(c) Determine whether $\tan 15^\circ$ is a constructible number.

Q5. Let A be an $n \times n$ matrix with entries in $\mathbb{C}$. Prove that A is conjugate to A^T, i.e., there exists an invertible matrix P such that $P^{-1} A P = A^T$.

Q6.

(a) Give a careful statement and proof of Eisenstein's irreducibility criterion for a polynomial in $\mathbb{Z}[x]$.

(b) Prove that the polynomial

$$x^{10} + x^9 + \cdots + x^2 + x + 1$$

is irreducible over $\mathbb{Q}$. (Hint: Use Eisenstein's criterion after a change of variable. However, a correct proof by any other method is also acceptable.)