Analysis Comprehensive Examination

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This examination contains three parts, Part A, Part B and Part C. Part A and Part B cover the core material described in unit I of the Analysis Comprehensive Syllabus; Part C covers the specialized material described in units II.(b) and II.(c) of the syllabus. The total time of the examination is six hours. Part A has six questions worth 10 points each, and you must attempt all questions in this part for a total possible score of 60 points. Part B has four questions worth 10 points each, of which you must attempt two, for a total possible score of 20 points. Part C has six questions worth 15 points each, of which you must attempt four, for a total possible score of 60 points. If you attempt in Part 2 or Part 3 more than the required number of questions, you must clearly indicate which questions are to be graded. If it is not clearly indicated, solutions to those appearing early in the booklet will be graded. You need to achieve at least 105 points, which is 75% of the total 140 attemptable points on the three parts, in order to pass the examination. No text or reference books, notes, calculators or aids are allowed in the exam.

Part A

Solve all the problems in this part.

1. Determine whether the statement is true or false. Justify your answer.

   (a) If $f : [0,1] \mapsto \mathbb{R}$ is absolutely continuous on $[0,1]$, then f is a function of bounded variation.

   (b) If $f : [0,1] \mapsto \mathbb{R}$ is continuous on $[0,1]$, differentiable on $(0,1)$, and such that f' is continuous on $(0,1)$, then f is a function of bounded variation.

2. Is it true that the series
$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n} \sin\left(1 + \frac{x}{\ln n}\right)$$
is uniformly convergent on $[-1,1]$. Justify your answer.

3. Evaluate the following limit:
$$\lim_{x \to 0} \frac{1}{x} \int_0^x \left(1 - \sin \frac{1}{u}\right) du.$$

4. Let Γ be the unit circle $|z| = 1$ with the positive orientation. Find
$$\int_\Gamma \bar{z} \frac{e^{2z}}{z^3} dz.$$

5. Evaluate the following integral using residue theory:
$$\int_{-\infty}^{\infty} \frac{dx}{x^2 - i}.$$

6. Let S be a dense subset of $\mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}$. Show that f is measurable if and only if the set $\{x \in \mathbb{R},\ f(x) \geq a\}$ is measurable for each $a \in S$.

Part B

Choose 2 from the following 4 problems to solve.

1. Evaluate the following line integral:

$$\oint_C \big(zy\sin(xy) + (x+y)^2\big)\,dx + \big((x+y)^2 + zx\sin(xy)\big)\,dy + \big(yz^3 - \cos(xy)\big)\,dz,$$

   where C is the curve of intersection of surfaces $z = \sqrt{x^2+y^2}$ and $(x-1)^2 + y^2 = 1$, directed counterclockwise when viewed from above.

2. Suppose that $0 < p < \infty$, and f is a continuous real valued function on $[a,b]$, where $a, b \in \mathbb{R}$. Is it true that, for every $\varepsilon > 0$, there exists an algebraic polynomial P with rational coefficients such that

$$\left(\int_a^b |f(x) - P(x)|^p dx\right)^{1/p} < \varepsilon.$$

3. Recall that Legendre polynomials form an orthogonal set with respect to the inner product $(f,g) = \int_{-1}^{1} f(t)g(t)dt$. The first four Legendre polynomials (normalized so that their values at 1 are 1) are (you do not have to prove this):

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \end{aligned}$$

   Find $a_0, a_1, a_2, a_3 \in \mathbb{R}$ such that the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is the polynomial of best approximation to $f(x) = x^4$ from the space of algebraic polynomials of degree ≤ 3 (in the $L_2[-1,1]$ norm).

4. Recall that a monotonic real-valued function defined on an interval is differentiable almost everywhere on the interval. Let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function. Show that

$$\int_0^1 f'(x)\,dx \leq f(1) - f(0).$$

   (You may assume without proof that $\int_0^1 f'$ exists.) Hint: Extend the definition of f to $[0,2]$ by defining $f(x+1) = f(1)$ for $x \in (0,1]$. Then consider

$$F_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}, \qquad n \geq 1.$$

Part C

Choose 4 from the following 6 problems to solve.

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and $E_1, E_2 \in \mathcal{A}$. Show that

$$\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2).$$

2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose that $(f_n) \subset L^1(\mu)$, $f \in L^1(\mu)$, and $f_n \to f$ μ-a.e. on X. Show that $\int |f_n - f| d\mu \to 0$ if and only if $\int |f_n| d\mu \to \int |f| d\mu$.

3. Let $(X, \mathcal{A}, \mu)$ be a finite measure space, $\mathcal{B}$ a sub-σ-algebra of $\mathcal{A}$, and $\nu = \mu|_{\mathcal{B}}$. Given $f \in L^1(\mu)$, show that there exists $g \in L^1(\nu)$ such that $\int_E f d\mu = \int_E g d\nu$ for all $E \in \mathcal{B}$.

4. Find a Möbius transformation that maps the triple $0, 1, i$ to, respectively, $2, -i, 1$.

5. Let G be an open set in the complex plane $\mathbb{C}$ and let $G = \cup_{n=1}^{\infty} K_n$, where $\{K_n\}$ is a sequence of compact sets satisfying $K_n \subset \text{int}(K_{n+1})$ for all n. Denote by $C(G)$ the complete metric space of all continuous complex-valued functions defined on G with the metric ρ given by

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f, g)}{1 + d_n(f, g)} \quad (f, g \in C(G)),$$

where $d_n(f, g) = \sup\{|f(z) - g(z)| : z \in K_n\}$ for each n. Let $H(G)$ be the subspace of $C(G)$ consisting of all analytic functions on G.

Suppose that $(f_m) \subset H(G)$ is a sequence and $f \in C(G)$ such that $\lim_{m\to\infty} f_m = f$ in the ρ-topology of $C(G)$. Show that $f \in H(G)$.

6. Suppose that G is an open connected set in $\mathbb{C}$. Show that a non-constant harmonic function u on G is an open mapping. (You may assume without proof that u is not a constant function on any nonempty open subset of G.)