

# Analysis Comprehensive Examination

October 28th, 2016

This examination contains three parts: Part One, Part Two and Part Three. The total time of the examination is six hours.

Part One has seven questions worth 10 points each, and you must attempt all questions for a total possible score of 70 points.

Part Two has three questions worth 15 points each, of which you must attempt two, for a total possible score of 30 points.

Part Three has three questions worth 15 points each, of which you must attempt two, for a total possible score of 30 points.

In Part Two and Part Three, if you attempt all three questions, you must clearly indicate which two questions are to be graded. If it is not clearly indicated, the first two questions appearing in the solutions will be graded.

You need to achieve at least 97.5 points (which is 75% of the total 130 possible points on the three parts) in order to pass the examination.

No text or reference books, notes, calculators or aids are allowed in the exam.

## PART ONE

1. If both functions  $f, g : [a, b] \rightarrow \mathbb{R}$  are of bounded variation on  $[a, b]$ , show that the function

$$h(x) := \max\{f(x), g(x)\}, \quad x \in [a, b],$$

is of bounded variation on  $[a, b]$ .

2. Let

$$f_n(x) = \frac{e^{-n^2x^2}}{n} \quad \text{for } x \in \mathbb{R}, \quad n = 1, 2, \dots$$

Prove that  $f_n \rightarrow 0$  uniformly on  $\mathbb{R}$ , that  $f'_n \rightarrow 0$  pointwise on  $\mathbb{R}$ , but that the convergence of  $\{f'_n\}$  is not uniform on any open interval containing the origin.

3. Using Stokes' Theorem, evaluate

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}, \quad \text{where } \mathbf{F}(x, y, z) = x\mathbf{i} - 2xy^2\mathbf{j} + xyz\mathbf{k},$$

and  $S$  is the part of the paraboloid  $z = x^2 + y^2$  bounded by the plane  $z = 1$ , oriented so that the unit normal vector points downwards (in the negative direction of  $z$ -axis).

4. Let  $I$  be an interval,  $\{\phi_n\}_{n=0}^\infty$  and  $\{\psi_n\}_{n=0}^\infty$  be two orthonormal systems from  $L^2(I)$  such that

$$\left\{ \sum_{n=0}^k \alpha_n \phi_n(x) : \alpha_n \in \mathbb{R}, k \geq 0 \right\} = \left\{ \sum_{n=0}^l \beta_n \psi_n(x) : \beta_n \in \mathbb{R}, l \geq 0 \right\}$$

(the set of all finite linear combinations of functions from the first system coincides with the set of all finite linear combinations of functions from the second system). Suppose  $f \in L^2(I)$  has expansions

$$f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x) \quad \text{and} \quad f(x) \sim \sum_{n=0}^{\infty} d_n \psi_n(x)$$

with respect to the first and the second system respectively. If

$$\lim_{m \rightarrow \infty} \left\| f - \sum_{n=0}^m c_n \phi_n \right\|_{L^2(I)} = 0,$$

prove that

$$\lim_{m \rightarrow \infty} \left\| f - \sum_{n=0}^m d_n \psi_n \right\|_{L^2(I)} = 0.$$

5. Determine whether or not the integral  $\int_1^\infty \frac{\sin x}{\sqrt[3]{x}} dx$  exists as:
- an improper Riemann integral,
  - a Lebesgue integral.

6. (a) Let

$$T(z) = \frac{z - a}{1 - \bar{a}z}.$$

Show that if  $a \in \mathbb{D} = \{z : |z| < 1\}$  then  $|T(z)| < 1$  for all  $|z| < 1$ . *Hint:* first show that if  $|z| = 1$  then  $|z - a|^2 = |1 - \bar{a}z|^2$ .

(b) Prove that every one-to-one and onto complex analytic mapping  $f$  from  $\mathbb{D}$  to itself is of the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

where  $a \in \mathbb{D}$ . *Hint:* let  $a$  be the point such that  $f(a) = 0$ , and use Schwarz lemma.

7. Evaluate

$$\frac{1}{2\pi i} \int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta}.$$

*Hint:* set  $z = e^{i\theta}$  and convert the integral to a contour integral.

## PART TWO

8. Let  $\mathcal{H}$  be a Hilbert space and let  $T : \mathcal{H} \rightarrow \mathcal{H}$  be a linear operator. Assume that there is another linear operator  $S : \mathcal{H} \rightarrow \mathcal{H}$  with the property that

$$\langle T\xi, \eta \rangle = \langle \xi, S\eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

Show that  $T$  is continuous with respect to the norm topology of  $\mathcal{H}$ .

9. Let  $(X, \mathfrak{A}, \mu)$  be a measure space, where  $\mu$  is  $\sigma$ -finite. Fix  $1 \leq p < \infty$  and let  $q$  be its conjugate, so that  $1/p + 1/q = 1$ . Let  $M \subset L^p(X, \mathfrak{A}, \mu)$  be a subspace and let  $\Phi$  be an element of the dual space  $M^*$ . Show that there is a function  $g_\Phi \in L^q(X, \mathfrak{A}, \mu)$  with  $\|g_\Phi\|_q = \|\Phi\|$  and such that

$$\Phi(f) = \int_X f g_\Phi d\mu, \quad f \in M.$$

10. Let  $X, Y$  be normed spaces and let  $M \subset X$  be a closed subspace. Let  $T : X/M \rightarrow Y$  be a bounded linear operator. Let  $q : X \rightarrow X/M$  be the quotient map. Show that  $\|T \circ q\| = \|T\|$ .

## PART THREE

11. (a) Let  $(X, \mathfrak{A}, \mu)$  be a measure space and  $f$  be an integrable function on  $X$ . Prove that  $\mu\{x \in X : |f(x)| = \infty\} = 0$ .
- (b) Let  $(f_n)$  be a sequence of extended real-valued measurable functions defined on  $X$ . Prove that if the series

$$\sum_{n=1}^{\infty} \left( \int_X |f_n| d\mu \right)$$

converges, then the series  $\sum_{n=1}^{\infty} f_n(x)$  converges  $[\mu]$  a.e..

12. (a) Let  $(X, \mathfrak{A}, \mu)$  be a measure space and  $(E_n)$  be an ascending sequence of measurable sets (i.e.  $E_n \subseteq E_{n+1}$  for all  $n$ ). Prove that

$$\mu(\cup_{n=1}^{\infty} E_n) = \lim_n \mu(E_n).$$

- (b) Let  $f$  be a non-negative measurable function on the space  $X$  above and  $m$  be the Lebesgue measure. Prove that

$$\int_X f d\mu = (\mu \times m)\{(x, y) : x \in X, 0 < y < f(x)\}.$$

13. Let  $(X, \mathfrak{A}, \mu)$  be a measure space and  $f$  be a non-negative measurable function on  $X$ .
- (a) Let  $v$  be a set-function defined by

$$v(E) = \int_E f d\mu, \quad (E \in \mathfrak{A}).$$

Prove that  $v$  is a measure on  $\mathfrak{A}$ .

- (b) Prove that for every non-negative measurable function  $g$  on  $X$ ,

$$\int_E g dv = \int_E f g d\mu \quad (E \in \mathfrak{A}).$$