Combinatorics Comprehensive Examination
Fall 2018

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9:00 AM Tuesday 30 January 2018
DRAFT: Feb 6 (key) AM

This exam has two parts, A and B. Answer all 10 questions from part A. Answer at most 5 of the 8 questions from part B, clearly indicating which questions you want graded. Each problem in part A is worth 5 points. Each question in part B is worth 10 points. The total possible score for this test is 100 points. A pass is 75/100. The total time for this exam is 6 hours.

No outside references or electronic aids are allowed.

Unless otherwise noted, all variables are non-negative integers and all graphs are simple. The vertex set of a graph G is denoted by $V(G)$ and the edge set is denoted by $E(G)$.

Part A

Do all problems. Each problem is worth 5 points.

A1. Find an explicit formula for a_n, where the sequence $a_0, a_1, \ldots$ has generating function

$$f(x) = \frac{7 - 5x}{2 - 3x + x^2}$$

A2. A school teacher has a class of 21 children, and she buys a bag of candies intending to give each student a candy. There are three kinds of candies: chocolate, vanilla, and strawberry.

(a) How many different ways can she give one candy to each student (assume there are sufficiently many candies of each type)?

(b) She polls the class to find out each child's preference, then brings exactly that many candies of each type the next day. How many different combinations of candies might be requested?

(c) Through polling each student, she discovers there are 11 children who like chocolate, 13 who like vanilla and 15 who like strawberry. There are 8 children who like both chocolate and vanilla, 8 who like both chocolate and strawberry, and 9 who like both vanilla and strawberry. Every child likes at least one kind of candy.

i. How many children like all three kinds of candies?

ii. The school teacher discovers that she only has 4 strawberry candies, does she have enough for each child who likes strawberry but no other kind of candy?

A3. For any graph G, and vertices $u, v \in V(G)$, let $d(u, v)$ be the distance between u and v. The **eccentricity** of a vertex $v \in V(G)$ is $\varepsilon(v) = \max_{w \in V(G)} d(v, w)$. The **center** of a graph is the subgraph induced by vertices of mimimum eccentricity. Prove that the center of a tree is a single vertex or the two endpoints of an edge.

A4. Prove that the edges of a graph G may be oriented so that the resulting digraph is strongly connected if and only G is connected and has no bridges.

A5. (a) Prove that, for any positive integer n,

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

(That is, for n odd, $\binom{n}{0} + \cdots + \binom{n}{n-1} = \binom{n}{1} + \cdots + \binom{n}{n}$
and for n even, $\binom{n}{0} + \cdots + \binom{n}{n} = \binom{n}{1} + \cdots + \binom{n}{n-1}$)
(You may assume the binomial theorem without proof.)

(b) Use a combinatorial argument to prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots \binom{n}{n}^2 = \binom{2n}{n}$$

A6. Derive a generating function $f(x) = \sum_{n\geq 0} h_n x^n$, in *closed form*, for the sequence $h_1, h_2, \ldots$, whose nth term is the number h_n of valid expressions of the form

$$b_1 + b_2 + \cdots + b_m = n$$

where $b_1, \ldots, b_m$ are odd positive integers and $m \geq 1$. (HINT: First solve for fixed m)

A7. Derive a formula for the number of ways to tile a $2 \times n$ rectangle with 1×2 tiles (which can be used in either orientation), which are identical except that they are coloured either red or blue. Show all steps in your derivation.

A8. State and prove the standard upper bound on the number of MOLS of order n for arbitrary n.

A9. Prove that, if the points of the plane are coloured red or blue there exists some rectangle that is monochromatic (which means that all four vertices are the same colour).

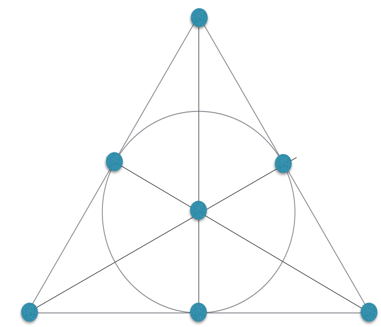
A10. Recall that a group is a set S with an associative operation $*$, an identity element e and every element a is invertible (there exists b such that $a * b = b * a = e$). Is the following Latin Square a group table or not? Give reasons for your answer.

$*$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	0	3	4	2	6	5
2	2	6	0	1	5	3	4
3	3	5	6	0	1	4	2
4	4	2	5	6	0	1	3
5	5	3	4	2	6	0	1
6	6	4	1	5	3	2	0

Part B

Do any five problems. Clearly indicate which problems are to be graded. Each problem is worth 10 points.

B1. Prove that if a finite projective plane $PG(2,k)$ contains the 7-point Fano plane, then $GF(k)$ is a field of characteristic 2



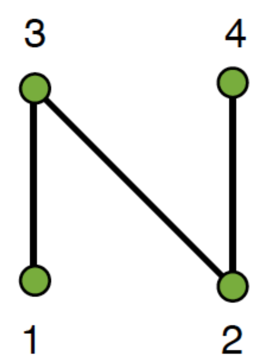
Fano Plane

B2. State Cayley's Theorem and prove it. While it is recommended to define and use Prüfer sequences to set up a certain one-to-one correspondence, any correct proof will be accepted as long as it is clear and fully justified.

B3. State and prove Dirac's Theorem about Hamilton cycles

B4. Define a **Steiner Triple System** (STS) on n elements. Also define a **Kirkman Triple System** (KTS) on n elements. Prove that if there is a STS on n elements then $n \equiv 1$ or $3 \pmod 6$. Prove that if there is a KTS on n elements then $n \equiv 3 \pmod 6$.

B5. Let N be the poset defined by the Hasse diagram shown. A set S is called a down-set if whenever y is in S then all the elements $\leq y$ are also in S. Find the lattice of all down-sets of N. Identify N as a subposet of the down-set. Prove that the lattice of all down-sets of N is distributive.



B6. Prove in a sequence of length $mn+1$ there is an increasing subsequence of length $m+1$ or a decreasing subsequence of length $n+1$.

B7. Let Π be a projective plane. We say Π_1 is a subplane of Π if the point set of Π_1 is a subset of the point set of Π, and the lines of Π_1 are lines of Π restricted to the point set of Π_1, and Π_1 satisfies the axioms of a projective plane.

Suppose that Π is a finite projective plane of order m and it contains a subplane Π_1 of order n where $n^2 = m$. Show that every line of Π contains at least one point of Π_1.

B8. (a) Prove Euler's formula $V - E + F = 2$ for a planar representation of a connected planar graph.

(b) Use part (a) to show that if G is a connected planar graph with 3 or more vertices ($V \geq 3$), then $E \leq 3V - 6$. Show K_5 is not planar.

(c) Use part (a) to show that if G is a connected planar graph with 3 or more vertices and has no triangles, then $E \leq 2V - 4$. Show $K_{3,3}$ is not planar.