UNIVERSITY OF MANITOBA

COMPREHENSIVE EXAMINATION

DATE: October 17, 2015 TIME: 6 hours

EXAMINATION: DE EXAMINER: DE Comprehensive Committee

INSTRUCTIONS TO STUDENTS:

This is a 6 hour examination. **No extra time will be given.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page and 8 pages of questions. Please check that you have all the pages.

Question	Points	Score
1	10	
2	10	
3	15	
4	15	
5	15	
6	10	
7	10	
8	15	
9	15	
10	15	
Total:	130	

The value of each question is indicated in the left margin beside the statement of the question.

The total value of all questions is 100 points, with 50 marks on each of Ordinary and Partial Differential Equations (ODE and PDE). The passing mark is 75 marks (75% of the total 100 points).

Each part (ODE and PDE) has the following structure, for a subtotal of 50 marks:

1. Two exercises worth 10 marks each and one exercise worth 15 marks, for a total of 35 marks. All these exercises are mandatory.
2. One exercise to be chosen from a list of two. The student will clearly indicate which of the attempted exercises they want marked. This exercise is worth 15 marks.

Ordinary differential equations

This part (ordinary differential equations) of the examination consists of 3 mandatory questions (questions 1-3) worth a total of 35 marks and one question worth 15 marks to be chosen from questions 4 and 5.

Please detail carefully your work.

Mandatory ODE questions: Answer all of the following three questions.

[10] 1. Consider the following initial value problem

$$y'' + 4y = g(t), \quad y(0) = 1, \quad y'(0) = 1.$$

where $g(t)$ is a piecewise continuous function with exponential order.

(a) Prove that the initial value problem has a unique solution on $\mathbb{R}$.

(b) Solve the initial value problem by using the Laplace Transform method.

[10] 2. Consider the equation $\dfrac{dx}{dt} = \lambda x - x^3$ where $x(t), \lambda \in \mathbb{R}$.

(a) Draw the bifurcation diagram.

(b) Determine the omega limit set $\omega(x_0)$ for $x_0 \in \mathbb{R}$.

[15] 3. Consider the following system

$$\frac{dx}{dt} = y + \alpha x(x^2 + y^2), \tag{1a}$$

$$\frac{dy}{dt} = -x + \alpha y(x^2 + y^2), \tag{1b}$$

(a) Find the equilibrium/ia of the system.

(b) Using the linearization method, what can you conclude about the equilibrium/ia found previously?

(c) Consider the function $V(x, y) = x^2 + y^2$. Using Lyapunov's method and the function $V(x, y)$, conclude about the stability of equilibrium/ia previously found.

Choice ODE question: Answer one (1) of the following two questions. Please indicate clearly which question you want marked.

[15] 4. Consider the system $\frac{dx}{dt} = Ax$ for a 3×3 real matrix A and $x(t) \in \mathbb{R}^3$. Suppose that $\det[A - \lambda \mathbf{I}] = (\lambda^2 + 1)(\lambda + 1)$. Let x_0 be an arbitrary point in $\mathbb{R}^3$. Show that the omega limit set of x_0, $\omega(x_0)$, is either the origin or a periodic orbit.

[15] 5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and suppose that for every t, the function $f(t, x)$ is non-increasing in x, that is, $f(t, x) \leq f(t, y)$, when $x > y$.

(a) Let $\psi_1(t)$ and $\psi_2(t)$ be two solutions of $\frac{dx}{dt} = f(t, x)$ defined on an open interval I. Show that if $\psi_1(\tau) = \psi_2(\tau)$ for some $\tau \in I$, then $\psi_1(t) = \psi_2(t)$ for $t \geq \tau$.

(b) Show that

$$f(t, x) = \begin{cases} |x|^{1/2} & \text{for} \quad x < 0, \\ 0 & \text{for} \quad x \geq 0, \end{cases}$$

satisfies the above condition and the initial-value problem

$$\frac{dx}{dt} = f(t, x), \quad x(0) = 0,$$

does not have a unique solution.

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}$, $s > 0$
2. e^{at}	$\frac{1}{s-a}$, $s > a$
3. t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}$, $s > \lvert a\rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}$, $s > \lvert a\rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}$, $s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}

Partial Differential Equations

This part (partial differential equations) of the examination consists of 3 mandatory questions (questions 6-8) worth a total of 35 marks and one question worth 15 marks to be chosen from questions 9 and 10.

Notation.

-For $x \in \mathbb{R}^N$ we set $|x| := \left(\sum_{k=1}^N x_k^2\right)^{\frac{1}{2}}$.
-For $z \in \mathbb{R}^N$ and $R > 0$ we set $B(z,R) := \{x \in \mathbb{R}^N : |x-z| < R\}$. We denote $B(0,R)$ by B_R.
- Given $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ we set $\Delta u(x) = \nabla^2 u(x) = \sum_{k=1}^N u_{x_k x_k}(x)$. In dimension $N = 2$ we will sometimes write $x = (x,y)$ instead of $x = (x_1, x_2)$.

Useful results.

- $\displaystyle\int_0^\pi \cos^2(kx)dx = \int_0^\pi \sin^2(kx)dx = \frac{\pi}{2}$ for all $k \geq 1$.
- (Integration by parts) Suppose Ω a smooth bounded domain in $\mathbb{R}^N$ and u, v smooth functions on $\overline{\Omega}$. Then

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega (-\Delta u) v dx + \int_{\partial\Omega} (\partial_\nu u) v dS,$$

  where $\partial_\nu u(x) = \nabla u(x) \cdot \nu(x)$ where $\nu(x)$ is the outward pointing normal on $\partial\Omega$ at $x \in \partial\Omega$.
- (Maximum principle)
  - Suppose $u \in C^2(\overline{\Omega})$ satisfies $-\Delta u(x) = f(x) \geq 0$ in Ω. Then

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{2}$$

  - Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + C(x)u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

    with $0 \leq C(x)$. Then $u = 0$ in Ω.

– Suppose $u \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -\Delta u + C(x)u &= f(x) \geq 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

with $0 \leq C(x)$. Then $u \geq 0$ in Ω.

Mandatory PDE questions: Answer all of the following three questions.

[10] 6. Let $B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and consider

$$u_{xx} + u_{yy} = 0 \qquad \text{in } B, \tag{5}$$

and suppose the boundary conditions, given in terms of polar coordinates, are $u(1,\theta) = \sin(4\theta)$. Find the solution u of (5) and you can write your answer in terms of polar coordinates, ie. $u = u(r,\theta)$. Note your answer need not involve an infinite sum.
Hint. Recall that $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}$.

[10] 7. Let $\Omega := \{x \in \mathbb{R}^N : |x| > 1\}$ where $N \geq 3$. Consider the following equation

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } |x| = 1. \end{cases} \tag{6}$$

(a) Find two solutions of (6). It may be helpful to recall that for a radial function $u(x) = u(|x|) = u(r)$ one has $\Delta u(r) = u''(r) + \frac{N-1}{r}u'(r)$.

(b) Find a solution of (6) which also satisfies $\lim_{|x|\to\infty} u(x) = 2$.

[15] 8. Consider a variant of the 'backwards heat equation' (where $u = u(x,t)$) given by

$$\begin{cases} u_t + 5u + u_{xx} &= 0 \qquad x \in (0,\pi),\ t > 0, \\ u(0,t) = u(\pi,t) &= 0 \qquad t > 0, \\ u(x,0) &= \phi(x) \qquad x \in (0,\pi), \end{cases} \tag{7}$$

where $\phi \in L^2(0,\pi)$.

(a) Using an infinite series approach write out a solution u of (7). You may leave integrals in your answer.

(b) Define

$$X := \left\{ \phi \in L^2(0, \pi) : (7) \text{ has a bounded solution } u, \right\}.$$

Compute X. You need not rigorously justify steps here; just some informal arguments are sufficient.
Hint. Note X is very simple.

Choice PDE question: Answer one (1) of the following two questions. Please indicate clearly which question you want marked.

[15] 9. Assume $N \geq 3$ and consider $A := \{x \in \mathbb{R}^N : 1 < |x| < 2\}$ where $|x|$ is the Euclidean norm of x. Suppose $\Omega \subset A$ and there is some $B(x^0, R) := \{x \in \mathbb{R}^N : |x - x^0| < R\} \subset \Omega$. We now suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\begin{cases} -\Delta u &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

and there are some positive constants K and M such that $K \leq f(x) \leq M$ for all $x \in \Omega$.

(a) (Upper bound on u) Find some function $v(x)$ defined in A such that $v = 0$ on ∂A and $v(x) \geq u(x)$ for all $x \in \Omega$. Use this function $v(x)$ to get an upper bound on $\sup_\Omega u$.
Hint. Recall that for a radial function $v(r)$ one has $\Delta v = v''(r) + \frac{N-1}{r} v'(r)$.

(b) (Lower bound on u) Find a positive lower bound on $\sup_\Omega u$.

[15] 10. Consider the heat equation given by

$$\begin{cases} u_t &= u_{xx} & x \in \mathbb{R},\ t > 0, \\ u(x,0) &= \phi(x) & x \in \mathbb{R}, \end{cases} \tag{9}$$

where $u = u(x,t)$ and ϕ is some given function. We will also consider the heat equation given on the half line by

$$\begin{cases} v_t &= v_{xx} & x \in (0,\infty),\ t > 0, \\ v(x,0) &= \psi(x) & x \in (0,\infty), \\ v(0,t) &= 0 & t > 0, \end{cases} \tag{10}$$

where the solution is $v = v(x,t)$ and $\psi(x)$ is some given function.

(a) Using the Fourier transform give a formula for a solution u of (9); your answer may have integrals in it. Here you must show some formal steps to arrive at the answer. (Need not rigorously justify the steps though).

(b) Let $v(x,t)$ denote the solution of (10) and write out the ODE satisfied by the Laplace transform (in t) of $v(x,t)$. Make it clear what is v and what is the transform. Also make it clear what quantities depend on which parameters. In your answer make sure there are no free functions or constants. Do **NOT** solve the ode or try and invert the Laplace transform.

(c) Write a formula for the solution v of (10), your answer may have integrals in it.
Hint. There is an easy way to do this using a suitable extension of ψ to $\mathbb{R}$ and then using your formula for the solution of (9). Justify (formally) that your solution does in fact satisfy the boundary condition at $x = 0$.