UNIVERSITY OF MANITOBA

PUBLISHED BY THE DEPARTMENT OF MATHEMATICS

MANITOBA MATH LINKS

BEAUTY DISCOVERY PRINCIPLES

Volume 5, Number 12, Winter 2005

A message from an applied math student

Darja Kalajdzievska
University of Manitoba

Although you might find yourself sitting in your math class thinking, "When am I ever going to use this?", mathematics is actually very relevant and useful in the real world.

Applied mathematics is a part of mathematics that describes objects and phenomena in the universe using numbers, equations and theorems. It can be used to analyze everything from the vibrations of a guitar string, to the crashing waves on the shore, to the flow of blood through the human body. It also translates into almost every field—biology, engineering, physics, management, economics, and even art.

Applied mathematicians are sought after in the work force. They do exciting jobs such as writing and breaking codes for the military, helping to send shuttles to space, and analyzing investment opportunities in the stock market.

Not only this, but there are also many jobs and research opportunities available for undergraduate students who are just beginning their studies. It is possible to find work assisting professors in research during the year and during the summers, and it is a much better work experience and pay than working at a *McDonald's* or a gas station. I can say this from personal experience. As a fourth-year applied math student focusing on mathematical problems in biology, I have been offered research work during the term by professors, and scholarships that last through the four months of summer have been my main source of income. I also find that this field of study is a great one for interacting with professors and other students, as the number of people studying applied math here at the University of Manitoba is relatively small. My experience with applied math has been so good that I plan to continue my studies toward a Master's degree after I graduate, even though many jobs require only an undergraduate degree.

Applied math is a great choice for someone who is curious about how the world around her or him works, and for someone who is innovative and likes to be challenged.

So, next time you're wondering why you need mathematics, think of this quote by James Caballero:

> "I advise my students to listen carefully the moment they decide to take no more mathematics courses. They might be able to hear the sound of closing doors."

Manitoba Math Links is a newsletter published three times a year (Fall, Winter, Spring) by the Outreach Committee of the Department of Mathematics, University of Manitoba. *Manitoba Math Links* gratefully acknowledges past financial support from *The Winnipeg Foundation* and assistance from the Manitoba Department of Education and Youth.

Submissions

Manitoba Math Links welcomes material on any topic related to mathematics, including articles, applications, announcements, humour, anecdotes, problems, and history. Materials are subject to editorial revision. Submissions may be made by regular mail or electronically to either the general address, or directly to any of the editors.
We also welcome editorial comments or suggestions from students, teachers, parents, or anyone interested in mathematics and mathematics education. The format of this newsletter is constantly evolving, so any input is appreciated.

Need more copies? Please feel free to reproduce and distribute this newsletter as needed; electronic copies (.pdf) may also be downloaded from our website, which is on the home page (see below) of the Department of Mathematics.

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SPEAKERS AVAILABLE

If you are interested in having a faculty member come to your school and speak to students, please contact the Department of Mathematics or any member of the editorial board.

Contents

Every human activity, except mathematics, must come to an end.

—*Paul Erdős (1913–1996)*

Quoted by Bela Bollobas, Amer. Math. Monthly **105** *(1998), 209.*

In the upcoming issue: π *is the limit*, a discussion of π and why the ratio of circumference to diameter is the same for any circle.

How many new mathematics teachers will Manitoba need next year? Ask Dr. Fermi!

Dr. Ralph Mason
Faculty of Education, U. of M.

In October, about 1000 Manitoba teachers attended the Manitoba Association of Mathematics Teachers (MAMT) Special Activity Group (SAG) Day. Imagine–one thousand math teachers in one building! I'm imagining something slightly differentI'm wondering if you have an experienced math teacher or a brand new one this year. Let me think about this. How many new math teachers are there in a year? I don't want to go look it up when I could just think about it. Well, since teachers retire after about 30 years, then we could assume that 1/30 of those thousand teachers will retire this year. That's 33 new math teachers we'll need. Some teachers quit; that might be (I wonder how we could estimate this proportion more accurately) another 1/30, or maybe another 1/10 of those teachers. I think I'll go with 2/30, to split the difference, so that's another 67 math teachers. So far, so good.

What other factors might affect the number of new math teachers we hire? Teachers go on parental leave, or study leave, too, but others will return from parental leave or study leave, so let's call that a draw. Temporary leaves are not going to make a large difference to the numbers of math teachers we need from year to year. Maybe if there are more students per class, there would be fewer classes and we'd need fewer teachers. Or maybe if more students chose to take more math (repeating PreCalculus 30S, maybe, or taking two different math classes in grade 11) than last year, we'd need more teachers. But I can't predict changes to those factors, so I'll assume that the total number of math teachers we need won't change.

Here's a possibility. Some teachers who teach math this year might switch to other subjects. But wait a minute. Other teachers might switch from other subjects into math, and I can't think of a reason that more would switch one way compared to the other. So that is another factor that won't affect the number I'm counting, but I'm glad I considered it. I know. What about math teachers who move up to roles as vice-principals or consultants? Not many teachers will move in the reverse direction, so this will mean there will be vacancies. Let's see: how many teachers might move up? If we think that there is one full-time administrator or consultant for every 10 teachers, then there would be 100 administrators for those 1000 math teachers. (There would be 100 administrators for every 1000 teachers of other subjects, too, presumably, but we'll only look at the ones that were teachers.) If people are administrators or consultants for, say, 10 years each on average, then 1/10 of those positions will be filled each year. 1/10 of 100 new administrators and consultants will mean 10 vacancies for new math teachers to fill. How are we doing? 33 + 67 + 10 = 110 new math teachers.

This kind of thinking is called *Fermi thinking*, named after Enrico Fermi, a famous physicist who liked to ask questions like, How many piano tuners do you think there are in New York City? Such questions give people a chance to exercise their quantitative reasoning, and to communicate their mathematical reasoning in convincing ways. In effect, it's a chance to practice proportional reasoning and mathematical justifications. If I wanted to really show off when I answer a question like the number of new math teachers that Manitoba needs, I could think about upper and lower limits for each element of my thinking. But in the thinking I shared above, I was not very fancy in my work; I was aiming for a quick and reasonable answer, and that's what I got.

It would be a different kind of task if someone really needed to accurately predict the number of new math teachers that Manitoba needs. We'd have to find more (and accurate) information before drawing our conclusions. (For example, the people who are deciding whether to use generalists or math specialists to teach grade 7 and 8 math are going to affect that number, so they would need more accurate information than this.) That would be a worthwhile task, but it's not Fermi thinking. Fermi thinking is not interested in accurate numbers; it's interested in first-quality mathematical reasoning.

But Fermi thinking is more fun to do than it is to read about. Here's one for you to try. And it will help you fix a big flaw in my own work, above. The number of math teachers in Manitoba is not going to be equal to the number of teachers who attend the MAMT SAG Day. It's much higher, for a variety of reasons we do not need to consider. Using that SAG Day number started me off somewhat badly. Instead, here's a better starting point of a different kind. According to the Winnipeg Free Press (2004 Sept 30, p. A2), the population of Manitoba on July 1, 2004 was 1,170,268 people. If you figure out how many of them are students, and how many math teachers we need for those students, you can use Fermi thinking to figure out how many math teachers there are in Manitoba.

If you give the question a try, I'd be interested in your strategies. I'm especially interested in how you choose to be efficient in your thinking: it could be a two-week research exercise if you forget the idea behind Fermi questions is to use estimation wisely to generate a reasonable and justifiable answer quickly!

If you send me a record of your thinking, I'll respond with some comments. My email is
`masonrt@ms.umanitoba.ca`.

> Mathematical rigor is like clothing; in its style it ought to suit the occasion, and it diminishes comfort and restricts freedom of movement if it is either too loose or too tight.
>
> —*G. F. Simmons*

Is that integer a perfect square?

R. G. Woods
Department of Mathematics, U of M

Is $k = 9,487,364,927,452$ a perfect square, that is, is there a whole number n such that $k = n^2$? Even using a calculator, you might find this tough to decide. Can we answer this question without using brute calculation?

Yes (in this case). There are some digits that cannot appear in the "units place" of a perfect square (the units place is the last digit, e.g., the last 2 in the above k.) Consider the following table of possibilities:

last digit of n	last digit of n^2	Reason	Example
0	0	$0 \times 0 = 0$	$40^2 = 1600$
1	1	$1 \times 1 = 1$	$221^2 = 73,441$
2	4	$2 \times 2 = 4$	$102^2 = 10,404$
3	9	$3 \times 3 = 9$	$83^2 = 6889$
4	6	$4 \times 4 = 16$	$254^2 = 64,516$
5	5	$5 \times 5 = 25$	$35^2 = 1225$
6	6	$6 \times 6 = 36$	$\vdots$
7	9	$7 \times 7 = 49$	get the
8	4	$8 \times 8 = 64$	idea?
9	1	$9 \times 9 = 81$	$\vdots$

Notice that there are four digits missing from the second column in the above table, namely 2,3,7, and 8. No perfect square can have any of these four digits in the units place! In particular, the integer k above has a 2 in its units place, so k is not a perfect square. That was easy! (By the way, notice the symmetry in the second column. What's the significance?)

Is the converse true? In other words, is every integer whose units digit appears in the second column a perfect square? No—consider 10, 11, 14, 19, 136, and 1,000,005 for example. So maybe we need some other tests for showing that an integer is not a perfect square.

If n is a positive integer and we divide it by 3, there are three possibilities—either n is divisible by 3, or the remainder is 1, or the remainder is 2. In other words, either n is of the form $n = 3m$, $n = 3m+1$, or $n = 3m+2$ for some non-negative integer m. Let's see what n^2 looks like in either of these three cases.

If $n =$	then n^2=	remainder upon dividing n^2 by 3
$3m$	$9m^2 = 3(3m^2)$	0
$3m+1$	$9m^2 + 6m + 1 =$	
	$3(3m^2 + 2m) + 1$	1
$3m+2$	$9m^2 + 12m + 4=$	
	$3(3m^2 + 4m + 1) + 1$	1

(Notice that if m is an integer, then any polynomial in m with integer coefficients is also an integer.)

We have just shown that if we divide n^2 by 3, then the remainder is never 2, or in mathspeak, "n^2 is never congruent to 2 modulo 3", no matter what the integer n is. For example, is 937,344,029 a perfect square? Dividing this number by three, $937,344,029 = 3(312,448,009) + 2$, the remainder is 2, and so 937,344,029 is not a perfect square.

Let's try the same thing using 4 instead of 3. If an integer n is not a multiple of 4, then there are three possible remainders, namely 1,2, and 3. Thus there is some integer m so that either $n = 4m$, $n = 4m+1$, $n = 4m+2$, or $n = 4m+3$. We get the following table of possibilities (like we did for 3 above).

If $n =$	then n^2=	remainder upon dividing n^2 by 4
$4m$	$16m^2 = 4(4m^2)$	0
$4m+1$	$16m^2 + 8m + 1 =$	
	$4(4m^2 + 2m) + 1$	1
$4m+2$	$16m^2 + 16m + 4=$	
	$4(4m^2 + 4m + 1)$	0
$4m+3$	$16m^2 + 12m + 9 =$	
	$4(4m^2 + 3m + 2) + 1$	1

Thus if we divide a perfect square by 4, the remainder is never 2 or 3. For example, though neither of the first two tests allows us to conclude that 120,031 is not a perfect square (the remainder upon division by 3 is 1). When dividing this number by 4, the remainder is 3, and so it cannot be a perfect square.

What we have done for 3 and 4 can be done for any other integer. You might check out for yourself what the possible remainders are when you divide a perfect square by 7. You might have noticed that in the first table that we produced, we were finding what the possible remainders are when you divide a perfect square by 10. If you do it, notice that the pattern in the list of possible remainders from division by 7 is similar to the pattern of remainders that we got by division by 10, but does not resemble the pattern of remainders that we got from 3 or 4. What is going on? Is there some underlying phenomenon that we have not identified?

One final fact: as you probably know, a prime number is an integer whose only integer divisors are 1 and itself [*Editor's note: Today, 1 is not considered a prime.*] It's a fact that every integer larger than 1 can be written in a unique fashion as a product of integer powers of primes. It immediately follows that an integer larger than 1 is a perfect square if and only if each of the exponents in this product is even.

Here is a challenge for you. What are the possible values for non-negative integers n and k (where $k \leq n$) so that when you divide n^2 by k, the remainder is $n-1$? See the next issue of *Math Links* for an answer.

> Mathematics is like checkers in being suitable for the young, not too difficult, and without peril to the state.
>
> —*Plato (427–347 B.C.)*

Cool Websites

R. Padmanabhan
Department of Mathematics

Mathematics in stamps

A postage stamp is, in a manner of speaking, a cultural ambassador of the country of issue. Philatelists all over the world take the postage stamp as an educational medium. They are attractive vehicles for conveying mathematical concepts and developments as well. Many countries around the world have celebrated significant mathematical achievements by issuing attractive commemorative postage stamps. In a recent publication on this topic (see [1]), Robin Wilson walks through 5000 years of mathematical history, visiting Egypt, Greece, China, India and even the Mayas and Incas. This is NOT a topic in the history of mathematics. It is, rather, a selective account of aspects of the history of mathematics that have appeared on postage stamps from across the world. Developments in Islamic mathematics, the middle ages and the rebirth of European mathematics are described. Wilson includes mathematically related inventions such as calendars, maps and globes on his way. Euclid, Newton (see [3]), Albert Einstein, Computers, prime numbers, Fermat's Last Theorem (shown above), Möbius band, mathematical instruments and fractals are some of the popular themes frequently occurring in postage stamps.

Impossible figures have proved to be irresistibly fascinating to artists, mathematicians and the stamp designers. In 1934, the Swedish artist Oscar Reutersvrd drew the first impossible triangle, an arrangement of nine cubes.

Three of his impossible figures were later featured in a set of Swedish stamps, issued in 1982 to commemorate his work (visit [6]). An exhibit of such a collage of postage stamps of various mathematical themes issued by countries around the globe will be an ideal Science Fair Project.

Apart from bringing mathematical ideas in pictures, such an exhibit will also demonstrate that mathematics, as a unifying theme, knows no geographical boundaries and will thus take us one step closer to the utopian concept of global village.

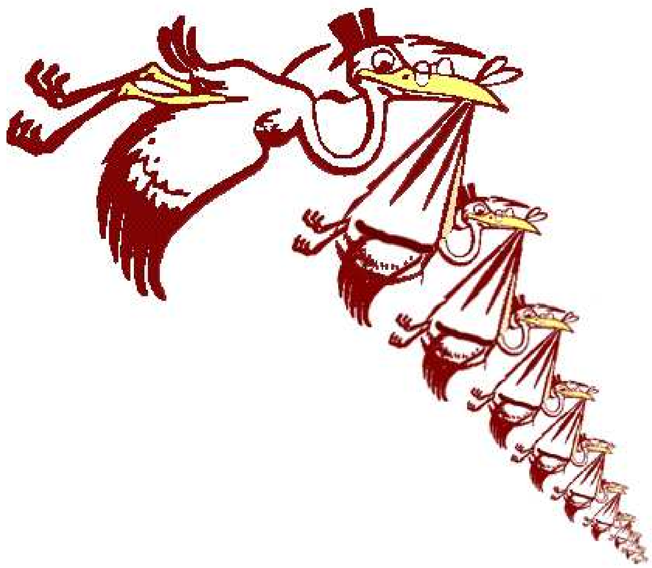
[1] Robin J. Wilson, *Stamping Through Mathematics*.
[2] http://www.oliver-faulhaber.de/mathstamps.htm#wilson
[3] http://www.geocities.com/newtonstamp/
[4] http://www.math.wfu.edu/ kuz/Stamps/stamppage.htm
[5] http://www.oliver-faulhaber.de/mathstamps.htm
[6] http://wwwhome.cs.utwente.nl/j̃agersaa/ Impossible.html
[7] http://www.math.ttu.edu/msu/philamath.html (this is a journal of mathematical philately).

Answers to Classic Puzzles: P1: one—the rest were coming *from* St. Ives! P2: On the 28th day, it begins at the 27 foot level, so goes up the remaining 3 feet and out! P3: Hint: first draw a right angle triangle covering seven dots not passing through the center, nor one of the upper corners. P4: Two out of three.

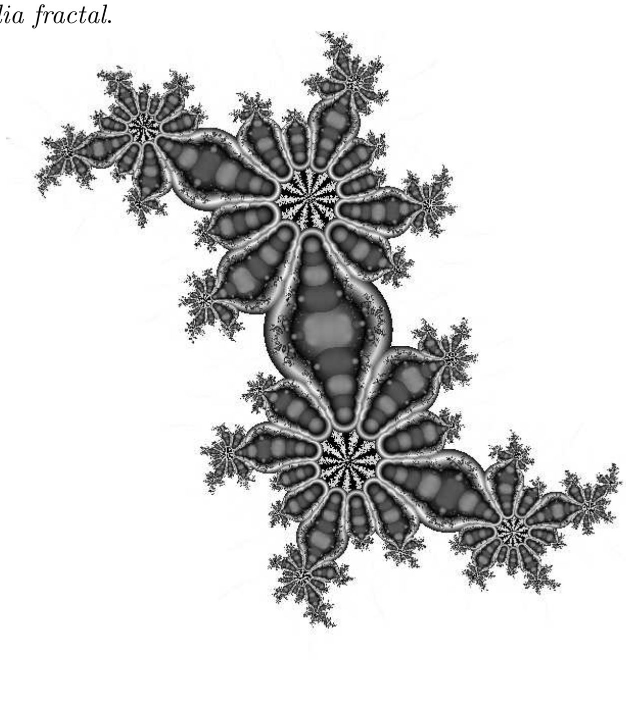
"Mommy, where do beautiful fractals come from?"

Sasho Kalajdevski
Department of Mathematics, U of M

"Well, storks bring them, my dear. But unlike baby-carrying storks, these ones carry smaller storks, and each smaller stork carries even a smaller stork, which in turn caries even a smaller stork, etc., continuing that without end. I made a picture of such a storks-carrying-storks for you to see."

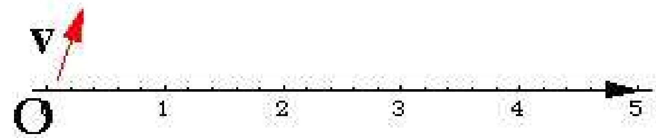


Fractals are like that: objects that are self-similar, and usually generated by some procedure that is carried out (iterated) infinitely many times. There are various types of fractals, some simple, some rather complicated. The stork-fractal shown above is relatively simple. Below we show one that is somewhat more complicated; we call it a *Julia fractal*.

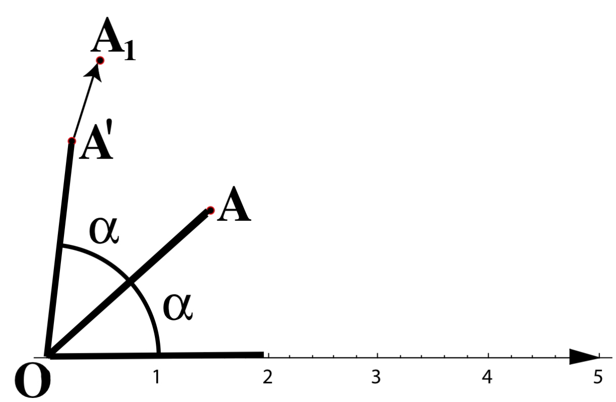


Computer-generated pictures of the same type as the Julia fractal we see here brought this subject to life, both from the mathematical and from the artistic point of view. Our goal in this note is to explain how this particular Julia fractal was generated.

Initially we have a semi-line (a ray) starting at the point denoted by O (origin), a unit distance that we mark on that semi-line, and a fixed vector (arrow), which we denote by $\mathbf{v}$. The vector $\mathbf{v}$ is chosen in the direction that goes 0.195875 units to the right and 0.5765 units upwards. You will notice how carefully we choose the vector: in this game small changes in the starting data may result in significant changes in the final object. The vector $\mathbf{v}$ and the fixed semi-line are shown in the next picture.

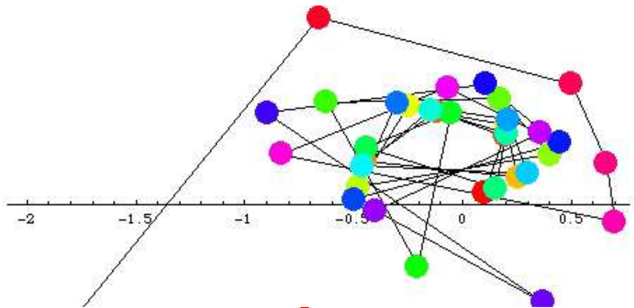


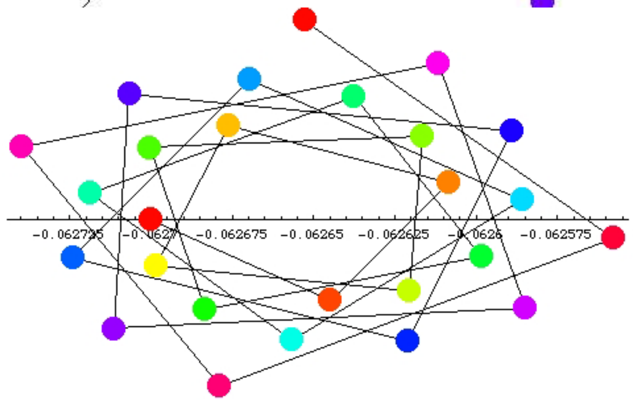
Now we move points in the plane according to the following prescription (we refer to the following picture below): first rotate the point A around O by the angle α and then move the newly obtained intermediate point A' in the direction of the vector v arriving at A_1.



Perform the same procedure to the new point: rotate A_1 around O by a new angle, the angle formed by the horizontal semi-line and OA_1, then move the new point A_2' in the direction of the vector $\mathbf{v}$, arriving at a point A_2. Keep doing this to get a sequence of points $A, A_1, A_2, A_3, \ldots$. Sometimes, the starting point A will generate a sequence of points that go farther and farther from the point O, while some other positions of the starting point A will generate a sequence of points that will hang around O. We illustrate these two possibilities in the next two pictures (we connect consecutive points in the sequences in order to track the movement of the points under our procedure).

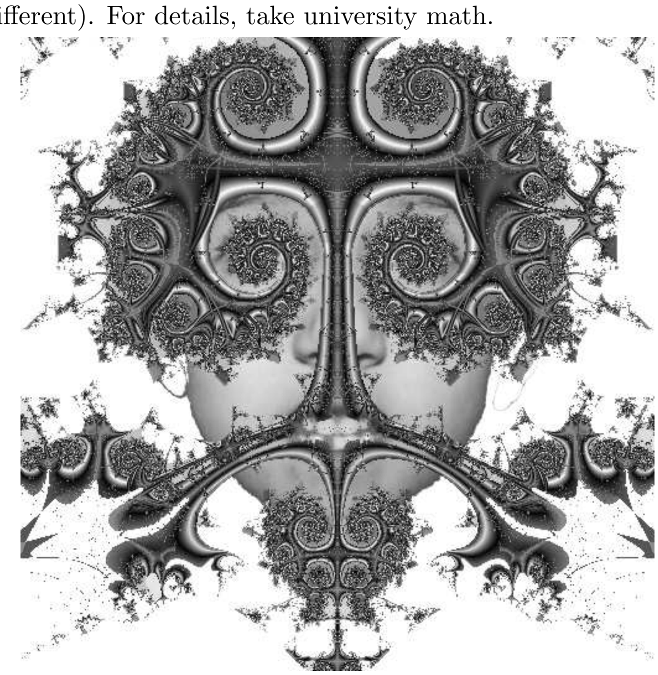
In the first picture, we see that at one moment the point takes off and goes far from O, while in the second picture the sequence of points hangs around keeping tight together. The starting points A that yield sequences of points that do no go far away from O is called the





prisoner set for our procedure, while the other starting points A that yield sequences of points that eventually go far away from O is called the *escape set* for our transformation.

Here is then the punch line: the boundary between the prisoner set and the escape set is what we have depicted in the Julia fractal shown above. We have described above a standard procedure for generating fractals: a specific movement of the points in the plane generated the boundary of the prisoner and the escape sets. Other movement will, of course, generate other prisoner/escape sets, and so, will produce other fractal pictures. One more example is given below: a mask-like fractal (that we superimpose over a photo of a girl) was obtained by following the same idea as above (but the movement of points in the procedure was different). For details, take university math.



Classic Puzzles

D. S. Gunderson
Department of Mathematics, U of M

First, the answer to the last classic puzzle: arrange the matches to make the figure 4.

The following four puzzles, many of which you might recognize all appear in Marcel Danesi's book, *The Puzzle instinct: the meaning of puzzles in human life* (Indiana University Press, 2002). Danesi has put together what looks like a scholarly work that discusses how puzzles seem to be central to the human psychology; I highly recommend it. More than just mathematical puzzles are discussed. It is interesting to note that many modern puzzles are just versions of older ones, some many thousands of years old!

An eighteenth-century popular nursery rhyme states:

> As I was going to St. Ives
> I met a man with seven wives.
> Each wife had seven sacks,
> Each sack had seven kits,
> Kits, cats, sacks, wives,
> How many were going to St. Ives?

Apparently, a version of this puzzle first appeared as Problem 79 in the Rhind Papyrus (written nearly 4000 years ago), and another was given by Fibonacci (in *Liber Abaci, 1202*), and since the papyrus was only found and deciphered in the 19th century, these versions were independent. The Fibonacci version had seven women, each with seven mules, each mule carrying seven sacks, each sack holding seven loaves, to slice each loaf there are seven knives, and for each knife, seven sheaths. How many are there altogether: women, mules, sacks, loaves, knives, sheaths? The answer is

$$7^1 + 7^2 + 7^3 + 7^4 + 7^5 + 7^6 = 137,256.$$

Here are four problems; answers appear on page 5 (try not to peek!).

> **P1:** How many are going to St. Ives?

Here is another from Fibonacci (that has found its way into modern puzzle folklore):

> **P2:** A snake is at the bottom of a 30-foot well. Each day it crawls up 3 feet and slips back 2 feet. At that rate, when will the snake be able to reach the top of the well?

The answer is a bit surprising! Here are two more classics:

> **P3:** Put nine dots in a 3 by 3 array. Using only three straight lines, draw lines through all nine dots.

Finally, one from Lewis Carroll:

> **P4:** A bag contains one counter, known to be either white or black. A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter?

Mathematical Partnerships 2004

Christine Ottawa
Mathematics Consultant
Winnipeg School Division

In 2004 a variety of mathematical opportunities occurred for students and teachers in the Winnipeg School Division as a result of collaborations with the Department of Mathematics at the University of Manitoba. These included:

- an information session for 25 high school mathematics teachers with faculty members from the Department of Mathematics organized and facilitated by Dr. Tom Berry
- a tour of the University of Manitoba for Senior 1 Mathematics students at Kelvin High School
- presentations by Dr. David Gunderson to Senior 1 students in the Enriched Mathematics course at Kelvin High School on Platonic Solids, the Golden Mean, Penrose Tiles, etc.
- participation by students in the 2004 Summer Math Camp at the University of Manitoba
- presentations by faculty members from the Department of Mathematics on enrichment topics to the students in the Calculus course at Sisler High School
- a copresentation by Dr. Gunderson and Christine Ottawa to high school mathematics teachers in the Winnipeg School Division on Mathematical Literacy: More Than Numbers and Words
- a presentation by Dr. Gunderson to Senior 1 students at Churchill High School on graphs and applications of graphs

These opportunities are welcomed as they provide perspectives and insights into the language of mathematics, the strategies necessary for solving problems in mathematics, the applications of mathematics, the thinking required in mathematics and the appreciation for mathematics. Sincere thanks is extended to Dr. Gunderson, Dr. Berry and faculty members from the Department of Mathematics for facilitating these exciting experiences in mathematics for the students and teachers in the Winnipeg School Division. We hope to continue these partnerships in 2005!

Problem corner

D. Trim
Department of Mathematics

Dear Readers:

Welcome once again to the PROBLEM CORNER. Here is the problem from the last column and its solution: Show that the second last digit of 2137^{753} cannot be even.

The last digits of 2137, 2137^2, 2137^3, 2137^4, and 2137^5 are 7, 9, 3, 1, and 7, respectively. Since $753 \equiv 1 \bmod 4$, it follows that the last digit of 2137^{753} is 7. Because $2137 \equiv 1 \bmod 4$, it also follows that $2137^{753} \equiv 1 \bmod 4$. Since the two-digit numbers 07, 27, 47, 67, and 87 are not congruent to 1 modulo 4, it follows that the second last digit of 2137^{753} cannot be even.

The Glenlawn Collegiate Math Club submitted a correct solution to the problem. They squared 37 and wrote down the last two digits. These they multiplied by 37 and once again wrote down only the last two digits. In effect, they were finding the last two digits of powers of 37 which would also be the last two digits of powers of 2137. They found that the last two digits of 37^{21} were once again 37, so that the last two digits would now cycle. Using modular arithmetic, they were then able to conclude that the last two digits of 2137^{753} are 97. Well done Glenlawn! You were perhaps fortunate that the cycle was so short; it could have been closer to 100. But not only did you prove that the second last digit could not be even as the problem required, you actually found it to be 9. Congratulations to you all. Perhaps this could be a challenge to other Math clubs to see who can submit correct solutions and who can do it first. Send submissions on the next problem to:

S. Kangas,
Department of Mathematics,
The University of Manitoba,
Winnipeg, MB
R3T 2N2

Here is your new problem: You are doing a jigsaw puzzle with 1500 pieces. Each day that you fit pieces together there are fewer pieces left, and therefore the puzzle becomes easier. Assuming that you can fit one extra piece each day compared to the day before, and you fit 20 pieces on the first day, on what day do you finish the puzzle and how many pieces do you fit on the last day?