UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

Graduate Comprehensive Exam in Topology

Friday April 24, 2015

14:00 to 19:00 (NO extra time will be given.)

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INSTRUCTIONS:

You have **five** hours to complete the exam.

The exam consists of this cover page and 5 pages of questions.

You must answer all seven (7) of the questions in Part A. For each of Part B and Part C you have a choice of questions. Answer any three (3) of the four (4) questions in each Part B and Part C.

For either Part B or Part C, you may attempt all four questions in that Part if you like, **but if you attempt all four questions appearing in that part, you must clearly indicate which answers are to be evaluated**. In the absence of any other indication, your solutions will be evaluated in the order that they are presented in your examination booklets.

To pass this exam, you must obtain a score of at least 70% in each of Parts A, B and C.

PART A Short answer questions

Answer all of Questions 1 through 8. The point values for each question are indicated in square brackets [ ].

[6] **Question 1.**

(a) Give the definition of a second countable space and a first countable space.

(b) Show that every second countable space is first countable.

(c) Give the definition of the subspace topology.

(d) Suppose that X is a second countable space. If $Y \subset X$ is given the subspace topology, show that Y is a second countable space.

[6] **Question 2.**
A topological space is called *totally disconnected* if every connected component is a singleton. Prove that any countably infinite metric space is totally disconnected.

[6] **Question 3.** Let X be a linearly ordered set. The topology generated by basis sets of the form $S_a = \{x|x > a\}$, together with X itself, is called the right order topology on X.

(a) Define what it means for a space to be each of the following: T_0, T_1, or T_2.

(b) Determine whether or not X with the right order topology is T_0, T_1, or T_2.

(c) Show that X is compact if and only if it contains a smallest element.

[6] **Question 4.**

(a) Define (in terms of open subsets in a space X) what it means to say that a set B is a dense subset of a space X.

(b) Prove that B is dense in X if and only if $\overline{B} = X$.

(c) Show that if B_j is dense in X_j, $j \in J$, then $\prod_{j\in J} B_j$ is dense in $\prod_{j\in J} X_j$.

[6] **Question 5.**

Set $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, and let $f : D \to D$ be a homeomorphism.

(a) Let L be the line segment connecting $A, B \in \partial D$, $A \neq B$. Show that $f(L)$ is not a subset of ∂D.

(b) Show that if $x \in \partial D$, then $f(x) \in \partial D$.

(c) Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

[6] **Question 6.**

(a) State the definition of homotopic maps.

(b) Let $f : X \to \mathbb{R} \setminus \{0\}$ be a continuous map. Find a mapping $g : X \to \mathbb{R} \setminus \{0\}$ such that $g(x)^2 = 1$ for all x and such that f is homotopic to g. [Hint: find g and then an explicit homotopy between f and g.]

[6] **Question 7.**

(a) State the definition of a *covering space* $(\tilde{X}, p)$ of a space X.

(b) Show that the covering mapping $p : \tilde{X} \to X$ is always open.

(c) Describe and depict a two-sheeted covering space of $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2 : 1 \leq x \leq 2\} \cup \{(x, y) \in \mathbb{R}^2 : (x - 3)^2 + y^2 = 1\}$.

PART B Point-set Topology

Answer 3 of Questions 8 through 11:

[10] **Question 8.**
Suppose that $A \subset X$. Let $f : A \to Y$ be a continuous map where Y is Hausdorff. If $g_1, g_2 : \overline{A} \to Y$ are continuous maps with $g_1|_A = g_2|_A = f$, then $g_1 = g_2$.

[10] **Question 9.** Let X be a non-empty set with the co-countable topology (i.e. non-empty open sets are complements of countable sets). Show that X is Lindelöf.

[10] **Question 10.**

(a) State Urysohn's Lemma and Tietze's Extension Theorem.

(b) Prove that Tietze's Extension Theorem implies Urysohn's Lemma.

[10] **Question 11.**

(a) Show that if X is Hausdorff, and if A_i, $i \in I$, are compact subsets of X, then $\bigcap_{i \in I} A_i$ is compact.

(b) Find a space X and two compact subsets of X such that their intersection is not compact.

PART C Algebraic Topology

Answer 3 of Questions 12 through 15:

[10] **Question 12.** Let $S^1 \vee S^1$ denote a bouquet of two circles (two circles joined at a single point). Show that there do not exist continuous maps $f : S^1 \vee S^1 \to S^1 \times S^1$ and $g : S^1 \times S^1 \to S^1 \vee S^1$ such that $g \circ f = id$.

[10] **Question 13.**
For a topological space X, the *cone* of X is the quotient $C(X) = (X \times [0,1])/\sim$ where $(x,0) \sim (x',0)$ for all $x, x' \in X$; while the *suspension* of X is the quotient $S(X) = (X \times [0,1])/\sim$ where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. Assume X is path connected.

(a) Show that $C(X)$ is simply connected. (Hint: In fact, it is contractible!)

(b) Show that $S(X)$ is simply connected. (Hint: Use Seifert-van-Kampen and use part (a).)

[10] **Question 14.** Let $(\tilde{X}, p)$ be a covering space of X.

(a) Show that the induced homomorphism $p_* : \pi_1(\tilde{X}) \to \pi_1(X)$ is always a monomorphism. [You may use the basic results on lifting paths.]

(b) Describe and depict a covering space $(\tilde{X}, p)$ of the torus $T^2 = S^1 \times S^1$ such that $p_*(\tilde{X}) \cong \mathbb{Z} \oplus \{0\}$, where $\mathbb{Z} \oplus \{0\} \subset \mathbb{Z} \oplus \mathbb{Z} \cong \pi_1(T^2)$

[10] **Question 15.**

(a) Compute the knot group of the trefoil knot (see the figure on the next page). Show your work.

(b) Let G_1 and G_2 be the knot group of knots K_1 and K_2, respectively. Suppose there is a plane in $\mathbb{R}^3$ that separates K_1 and K_2. Show that $\pi_1(\mathbb{R}^3 \setminus (K_1 \cup K_2)) = G_1 * G_2$.