$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ – The Wave Equation

Water Waves in the Ocean
Figure 1

Waves are all around us. We are all familiar with water waves and sound waves and know that waves on guitar strings produce music. Also, most of us have heard that electromagnetic radiation can be thought of as traveling in waves, however, we can't see these waves – we only see their effects as light, radio and TV programs, x-ray pictures or in using cell phones. There are also seismic waves, microwaves, shock waves and gravitational waves!

A wave can be defined as a disturbance that travels through space and time.

Most waves can be described by the wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$,

where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ in 3-dimensional space.

(x, y, z) represent width, depth and height and t represents time.
The function $u = u(x, y, z, t)$ represents the displacement of the wave.

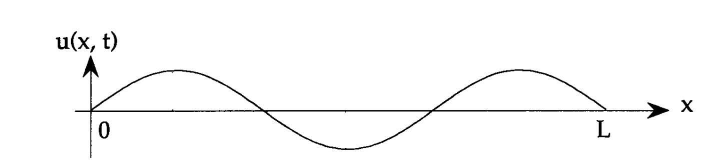

Figure 2

Consider the transverse vibrations of a string (as shown in Figure 2). The wave equation for the transverse displacement, $u(x, t)$, of this string is: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$. Here, $c^2 = \frac{T}{\rho}$, where T is the tension and ρ is the linear density of the string. The constant $c = \sqrt{T/\rho}$ is also the speed of the waves along the string. Let's look at this wave equation: The quantity $\frac{\partial u}{\partial t}$ is the rate of change of u with respect to time, t; this is known as the vertical *velocity* of a point on the string. The term $\frac{\partial^2 u}{\partial t^2}$ is the rate of change of the velocity, which is known as the vertical *acceleration*. The factor $\frac{\partial^2 u}{\partial x^2}$ is closely related to the *curvature* of the string (the larger this quantity, the more the string is curved or bent). Thus, the wave equation tells us that if the string is curved downward, then there will be a downward acceleration which is proportional (through c^2) to the curvature, and similarly for an upward acceleration corresponding to an upward curvature.

If the two end points are attached to the x-axis, then the solutions of this equation will be infinite sums of terms of the form: $a_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$ and $b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$, where a_n and

b_n are constants. These terms are called *standing waves* or the *fundamental modes of vibration.* Thus, we see that the solutions of the wave equation are described in terms of the trigonometric functions, sin and cos.

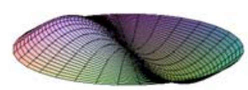

Figure 3

We now consider a vibrating membrane (i.e., a drum) (shown in Figure 3). The wave equation for the transverse displacement, u (x, t), of this drum is: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \right)$.

This equation also represents the ripples in a pond of water if we drop a stone at some point into the pond. (Then, this point will be at the centre of the ripples; see Figure 4, below).

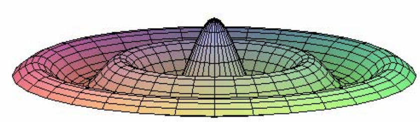

Figure 4

For these two cases, the situation is much more complicated than the string since the waves spread out in two dimensions! Under certain conditions, the solutions of this equation will be infinite sums of terms of the form: $k_n J_0(a_n r)\cos(a_n ct)$, where k_n and a_n are constants and r is the distance of a point from the centre of the drum or ripples. J_0 is a complicated function called the Bessel function of the first kind of order zero. Below is a plot of $J_0(x)$ for $-15 \le x \le 15$. It provides a cross-section of the wave shown in Figure 4.

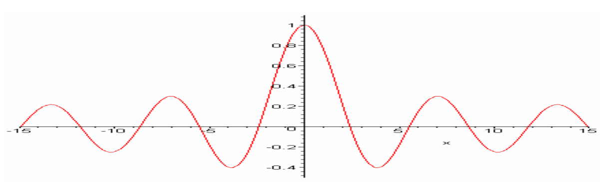

Figure 5

Figure 1: By Mila Zinkova at http://en.wikipedia.org/wiki/File:Waves_in_pacifica_1.jpg

Figures 2 to 5: Produced by Dr. Joseph J. Williams, Department of Mathematics, University of Manitoba