Combinatorics comprehensive, Fall 2017

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This exam has one part consisting of 15 questions. All questions should be attempted. There are ten problems worth 5 points each, and five problems worth 10 points each. The total possible score for this test is 100 points. A pass is 75/100. The time alloted for this exam is 6 hours.

No outside references or electronic aids are allowed. Unless otherwise indicated, the following notation is assumed.

Notation

G	a simple graph
$V(G)$	the vertex set of G
$E(G)$	the edge set of G
$\delta(G)$	the minimum degree of G
BIBD	balanced incomplete block design
SBIBD	symmetric balanced incomplete block design

Section 1: all of the following ten problems should be attempted. They are worth 5% each.

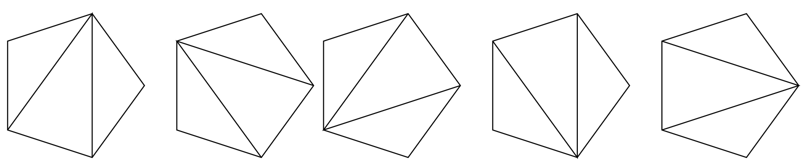
1. Let $S = \{1, 2, \ldots, n\}$.
   (a) Give the definition of a derangement of S.
   (b) Find a formula for the number of derangements of S.
   (c) What happens to the ratio of number of derangements of S to the number of permutation of S as n becomes large?

2. Prove that for $n \geq 3$, every orientation of the complete graph K_n has a directed Hamiltonian path. Show that the same is not true for Hamiltonian cycles.

3. Let G be a connected cubic graph with a cut vertex. What are the possible values of the edge connectivity?

4. A sequence x_n with initial conditions $x_0 = 0$ and $x_1 = 2$ satisfies
$$x_n = 4x_{n-1} - 3x_{n-2} \text{ for } n \geq 2.$$
   (a) Evaluate x_n in a closed form.
   (b) Let $f(t) = \sum_{k=0}^{\infty} x_n t^n$. Write $f(t)$ as a function in closed form.

5. Let $C_0, C_1, \ldots$ be the Catalan sequence.
   (a) How are the Catalan numbers defined recursively?
   (b) Find C_0, C_1, C_2, C_3, C_4.
   (c) A *triangulation* of a regular polygon P_n with n vertices is an insertion of nonintersecting edges joining vertices so that every face is a triangle. The five triangulations for P_5 are:



   Show that the number of triangulations of P_n is C_{n-2} for $n \geq 3$.
   (d) Let $c(t) = \sum_{k=0}^{\infty} C_k t^k$. Use the definition of Catalan numbers to find a relationship between $c(t)$ and $c(t)^2$.

6. (a) Define the chromatic number, denoted $\chi(G)$, of a graph G.
   (b) Let $k \geq 2$ and G be a graph on n vertices with minimum degree $\delta(G) > \frac{(k-2)}{(k-1)} n$. Prove that $\chi(G) \geq k$.

7. Let G be a graph with minimum degree $\delta(G) \geq 2$. Show that there is a connected graph H with the same degree sequence as G.

8. Prove that if there is an Hadamard design SBIBD$(4q-1, 2q-1, q-1)$ then there is an SBIBD$(8q-1, 4q-1, 2q-1)$.

9. Construct three mutually orthogonal 7×7 latin squares.

10. Give the definition of the **derived design** and the **residual design** of a BIBD, and illustrate with the design consisting of all 4-subsets of a 6-set. Prove that the residual and derived designs of a SBIBD(v, k, λ) are both BIBDs and give their parameters.

Section 2: all of the following five problems should be attempted. They are worth 10% each.

1. (a) What are the three properties of an equivalence relation?
   (b) For any pair of these properties, give an example of a relation satisfying those two properties but not the remaining one.
   (c) Let $S(n, k)$ be the number of equivalence relations on a set of n elements with exactly k equivalence classes. Find a recursive relation for $S(n + 1, k)$ in terms of $S(n, t)$, $t = 0, 1, \ldots n$.

2. (a) Define a partial order on a set X.
   (b) Define a chain and an antichain for a partial order X.
   (c) Suppose, for some partial order X, m is the minimum number of chains that covers X and M is the size of the largest antichain. Show that $m \geq M$.
   (d) Let $\mathcal{B}(n)$ be the set of all subsets of $\{1, 2, \ldots, n\}$. Show that the relation $\subseteq$ is a partial order on $\mathcal{B}(n)$.
   (e) For $\mathcal{B}(4)$, find the value of m and M. Prove that your values are correct.
   (f) Prove that $m = M$ for any partial order.

3. Suppose that $n \geq 1$, G is a bipartite graph on partite sets U, W with $|U| = |W| = n$, and G has at least $n^2 - n + 1$ edges. Prove that G has a perfect matching. Clearly state any theorems used in your proof.

4. Let G be a bipartite graph with partite sets U, W of size $|U| = |W| = n \geq 2$, and let the minimum degree satisfy $\delta(G) > n/2$. Prove that G is Hamiltonian.

5. (a) State the Bruck-Ryser-Chowla Theorem for (v, k, λ) designs.
   (b) Indicate what, if anything, this theorem says about the existence of projective planes of order n for $2 \leq n \leq 15$.
   (c) Prove the theorem for the case when v is even.