Introductory Numerical Analysis

This unit consists of four questions worth a total of 40 marks. Answer all questions.

1. (a) [4 points] Let f be a smooth function. Show that

$$f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - \frac{h^2}{12} f^{(4)}(\xi)$$

for some $\xi \in [x_0 - h, x_0 + h]$. State which theorems you have used in this proof.

(b) [4 points] Suppose that $|f^{(4)}(x)| \leq M$ for all $x \in [x_0 - h, x_0 + h]$. Suppose that all numerical evaluations of f have round off errors bounded by δ. Find an upper bound for the total error in approximating $f''(x_0)$ using the centered difference approximation $\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2}$.

(c) [2 points] Find the value of h (in terms of M and δ) that will minimize the total error bound.

2. (a) [5 points] Find the unique polynomial of degree at most four that interpolates $f(n^3) = 0$ for $n = -1, 0, 1, 2, 3$. (No simplification is necessary.)

(b) [5 points] Let $x_0 < x_1 < ... < x_n$ and $L_{n,k}(x)$ be the fundamental Lagrange polynomial corresponding to x_k. Prove that

$$\sum_{k=0}^{n} L_{n,k}(x) = 1.$$

3. [10 points] Consider the following initial value problem

$$y' = f(y), \qquad y(t_0) = y_0, \tag{*}$$

where $y_0 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Heun's method for the initial value problem is given by

$$\begin{aligned} w_0 &= y_0, \\ w_{i+1} &= w_i + \frac{h}{4}\left[f(w_i) + 3f\left(w_i + \frac{2}{3}hf(w_i)\right)\right], \end{aligned}$$

where h is the fixed stepsize. Give the local truncation error of this method. What is the order of the method?

4. Consider the initial value problem given in (*) and suppose that $y_0 \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. The backward Euler method for the system $y' = f(y)$ (for $y \in \mathbb{R}^n$) is given by

$$\begin{aligned} w_0 &= y_0, \\ w_{i+1} &= w_i + hf(w_{i+1}). \end{aligned}$$

where h is the fixed stepsize. Recall that Newton's method for finding the zero of a function $g : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is given by the iteration

$$x_{k+1} = G(x_k),$$

where $G(x) = x - J(x)^{-1}g(x)$ and $J(x)$ is the Jacobian matrix of g evaluated at x.

(a) [1 point] Given f, w_i and h, write down a function $g(x)$ such that the Backward Euler update w_{i+1} must satisfy $g(w_{i+1}) = 0$.

(b) [4 points] Consider the backward Euler method applied to the linear system $u' = Au$ where $A \in \mathbb{R}^{n\times n}$ and $u(0) = u_0 \in \mathbb{R}^n$. Suppose that h^{-1} is not an eigenvalue of A. Find an expression for the unique solution w_{i+1} to the backward Euler equation in this case.

(c) [5 points] Suppose that we solve for the backward Euler update w_{i+1} of the linear system in part (b) by Newton iteration. This means we set $w_{i+1} = \lim_{k\to\infty} x_k$ where $x_{k+1} = G(x_k)$ for some appropriate G. Find an expression for $G(x)$ in terms of A and show that the Newton iteration converges in one step to the solution in part (b).

Approximation Theory

This unit consists of five questions worth a total of 40 marks. Answer all questions.

1. [6 points] Prove that for any continuous $f : [0, \infty) \to \mathbb{R}$ that has a finite limit $\lim_{x\to\infty} f(x)$ and for any $\varepsilon > 0$ there exist positive integer n and real numbers a_k, $k = 0, \dots, n$, such that

$$\left| f(x) - \sum_{k=0}^{n} a_k e^{-kx} \right| < \varepsilon \quad \text{for all } x \geq 0.$$

2. [8 points] Prove that

$$E_n(x_+^\pi)_{C[-1,1]} \leq cn^{-3}, \quad n \geq 1,$$

where c is a positive absolute constant and $x_+ = \max\{x, 0\}$. (Here $E_n(f)_{C[-1,1]}$ denotes the error of the best uniform approximation of f on $[-1, 1]$ by algebraic polynomials of degree $\leq n$.)

3. [8 points] Using Bernstein's inequality, prove that for any trigonometric polynomial τ_n of degree $\leq n$, the inequality

$$\|\tau_n\|_{L_1([-\pi,\pi])} \geq \frac{c}{n} \|\tau_n\|_{C([-\pi,\pi])}$$

is valid, where c is some positive constant independent of n.

4. [7 points] Let $L(x, g)$ be the Lagrange interpolation polynomial of degree $\leq m$ interpolating a function $g \in C[0, 1]$ at the points $x_j = \frac{j}{m}$, $j = 0, \dots, m$, $m \geq 1$. Prove that

$$\|L(\cdot, g)\|_{C[0,1]} \leq (m + 1)^{m+1} \|g\|_{C[0,1]}.$$

5. [11 points] Using the usual Whitney's inequality, prove the following shape preserving version of this inequality: if $f \in C[0, 1]$ is convex on $[0, 1]$, then there exists a convex on $[0, 1]$ polynomial p of degree 2 satisfying

$$\|f - p\|_{C[0,1]} \leq c\omega_3(1, f, [0, 1])$$

with a positive absolute constant c. (Here $\omega_3(1, f, [0, 1])$ denotes the usual third order modulus of smoothness of f on $[0, 1]$ with step 1.) (Hints: choose p as an appropriate Lagrange polynomial; result of the previous problem may be useful.)

Numerical Analysis of PDEs

This unit consists of seven questions worth 10 marks each. Answer four questions. You may attempt as many questions as you like in this unit; however, if you attempt more than four questions, you must clearly indicate which answers you want us to mark. In the absence of any explicit indication, we will mark respectively the first four questions for this unit.

1. [10 points] For the Poisson equation

$$-\frac{1}{r}(ru_r)_r - \frac{u_{\theta\theta}}{r^2} = f$$

on the unit disk, approximate the equation by a second order finite difference scheme using polar coordinates $r_i = i\Delta r$, $\theta_j = j\Delta\theta$. Specify the discretization scheme at the origin.

2. [10 points] Analyze the stability of the Crank-Nicholson scheme for the heat equation

$$u_t = u_{xx} + f(x,t), \quad x \in (0,1), \quad t > 0$$

under the conditions

$$\begin{aligned} u(x,t) &= 0, \quad x = 0, \quad x = 1, \quad t > 0, \\ u(x,0) &= g(x), \quad x \in (0,1). \end{aligned}$$

3. [10 points] For the equation $u_{xx} + u_{yy} - 2u = f$ in Ω where Ω is a square, f is smooth and $u = 0$ on the boundary, derive the usual second order difference approximation. Find an expression for the consistency error $O(k^2)$.

4. [10 points] Develop a second order finite difference scheme for the equation

$$u_{tt} = u_{xx} + u, \quad x \in (0,1), \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = g(x), \quad u(0,t) = u(1,t) = 0.$$

Show that the explicit scheme is consistent and determine the stability if the solution lies in $C^4(\overline{\Omega}^T)$.

5. [10 points] Consider the PDE $-\Delta u + u = f \in L_2$, where $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Let V_h be the usual finite element subspace of $H_0^1(\Omega)$. Derive the weak form of the PDE and its finite element equations. Estimate the condition number of the stiffness matrix.

6. [10 points] For the Galerkin method discuss the solution of $Lu = f$ for $u \in V'$, a separable Hilbert space, where $f \in V'$. If the bilinear form is bounded and coercive, show the convergence of the Galerkin scheme.

7. [10 points] Use the Richardson iteration $x^{(n+1)} = x^{(n)} + \omega(b - Ax^{(n)})$, ω a real number, to solve the linear system $Ax = b$ where A has positive real eigenvalues. Starting with any initiall guess find values of ω for which the iteration converges. Also find the best value of ω which minimizes the spectral radius of the iteration matrix.