Linear Algebra Review

Matrices

A **matrix** is a rectangular array of numbers that are arranged in rows and columns. A matrix M is called $m \times n$ if it has m rows and n columns. The numbers m and n are the **dimensions** of M. Examples:

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 5 \\ -3 & 2 \\ 0 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 4 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix}.$$

In the above, A is 2×3, B is 3×2, C is 1×3, D is 2×2.

The entry in row i and column j of matrix M is denoted by M_{ij}. For example, $A_{23} = 5$, $B_{21} = -3$, $D_{12} = 2$.

Addition, subtraction, scalar multiplication

Two matrices M and N can be added or subtracted if and only if they have identical dimensions, $m \times n$. In that case, the sum or difference is also an $m \times n$ matrix, and

$$(M+N)_{ij} = M_{ij} + N_{ij}, \qquad (M-N)_{ij} = M_{ij} - N_{ij}.$$

For example,

$$\begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 6 & 5 \\ -3 & 2 \\ 0 & 7 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ -1 & -8 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -2 & 10 \\ -2 & 8 \end{pmatrix},$$

but $\begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 6 & 5 \\ -3 & 2 \\ 0 & 7 \end{pmatrix}$ does not exist.

Any matrix M can be multiplied by a scalar r. If M is $m \times n$, then rM is also $m \times n$, and

$$(rM)_{ij} = r(M_{ij}).$$

For example,

$$5\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix} = \begin{pmatrix} 15 & 5 & -5 \\ 0 & -10 & 25 \end{pmatrix}, \qquad -3\begin{pmatrix} 0 & 1 \\ -2 & 3 \\ 7 & -9 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -3 \\ 6 & -9 \\ -21 & 27 \\ -6 & 0 \end{pmatrix}.$$

Matrix multiplication

Given two matrices M and N, the product MN exists if and only if M is $m \times n$ and N is $n \times r$, in which case MN is $m \times r$. The entries in MN are given by

$$(MN)_{ij} = \sum_{k=1}^{n} M_{ik} N_{kj}.$$

For example,

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ -3 & 2 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 15 & 10 \\ 6 & 31 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 4 & 2 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ -3 & 2 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} -18 & 17 \end{pmatrix},$$

but $\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix}$ does not exist.

Transpose

Given an $m \times n$ matrix M, the **transpose** of M, notation M^T, is the $n \times m$ matrix given by

$$(M^T)_{ij} = M_{ji}.$$

That is, M^T is the result of interchanging the rows and columns of M. For example,

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix} \implies A^T = \begin{pmatrix} 3 & 0 \\ 1 & -2 \\ -1 & 5 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 2 \\ 1 & -3 \end{pmatrix} \implies D^T = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}.$$

If M and N are matrices such that the product MN exists, then

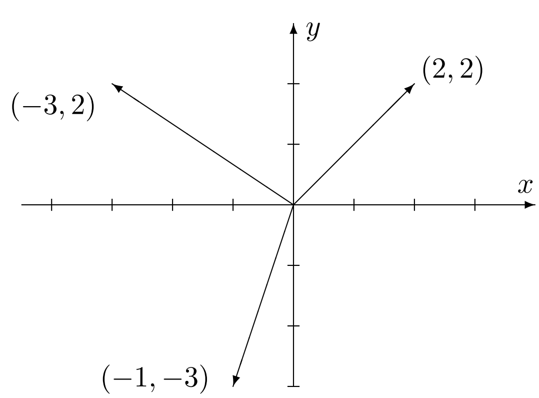
$$(MN)^T = N^T M^T.$$

Vectors

A vector in $\mathbb{R}^n$ is a quantity with magnitude and direction: that is, an arrow. It can be represented by an ordered n-tuple of real numbers, notation

$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

The entries x_j are the **components** of the vector $\mathbf{x}$. A vector with n components is often called an n-vector. Below are some examples of 2-vectors.



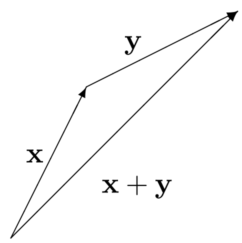
Vector addition, length, scalar multiplication, unit vectors

Two vectors $\mathbf{x}$ and $\mathbf{y}$ can be added or subtracted if and only if they have the same number of components, in which case

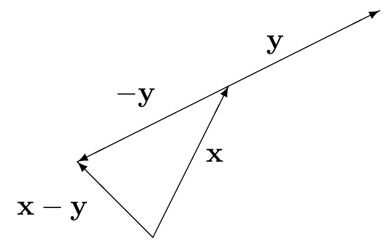
$$\mathbf{x} \pm \mathbf{y} = (x_1, \ldots, x_n) \pm (y_1, \ldots, y_n) = (x_1 \pm y_1, \ldots, x_n \pm y_n).$$

Geometrically, to add the n-vectors $\mathbf{x}$ and $\mathbf{y}$, we line them up head to tail. Then $\mathbf{x}+\mathbf{y}$ points from the tail of $\mathbf{x}$ to the head of $\mathbf{y}$. Subtracting the vector $\mathbf{y}$ is the same as adding the vector $-\mathbf{y}$.

Addition:



Subtraction:



The **length** of the n-vector $\mathbf{x}$, notation $|\mathbf{x}|$, is

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

Any vector $\mathbf{x}$ can be multiplied by a scalar r, and the result is given by

$$r\mathbf{x} = (rx_1, \ldots, rx_n).$$

It follows that

$$|r\mathbf{x}| = |r||\mathbf{x}|.$$

Geometrically, if $r > 0$, then $r\mathbf{x}$ has the same direction as $\mathbf{x}$, and its length is scaled by a factor of r. If $r < 0$, then $r\mathbf{x}$ has the opposite direction from $\mathbf{x}$, and its length is scaled by a factor of $|r|$.

Two nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are **parallel** if there is a scalar r such that $r\mathbf{x} = \mathbf{y}$.

If $\mathbf{x}$ is a vector such that $|\mathbf{x}| = 1$, then $\mathbf{x}$ is called a **unit vector**. If $\mathbf{y}$ is a vector such that $|\mathbf{y}| \neq 0$, then the unit vector in the direction of $\mathbf{y}$, notation $\widehat{\mathbf{y}}$, is given by

$$\widehat{\mathbf{y}} = \frac{1}{|\mathbf{y}|}\mathbf{y}.$$

Dot product, orthogonal vectors, projections

Let $\mathbf{x}$ and $\mathbf{y}$ be two n-vectors. The **dot product** (also called the **scalar product**) of two vectors $\mathbf{x}$ and $\mathbf{y}$, notation $\mathbf{x} \cdot \mathbf{y}$, is defined if and only if $\mathbf{x}$ and $\mathbf{y}$ are both n-vectors, in which case

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \ldots + x_ny_n.$$

Note that the dot product of two vectors is a *scalar*, not a vector. If $\mathbf{x}$, $\mathbf{y}$ $\mathbf{z}$ are n-vectors and r is a scalar, then

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} =& \mathbf{y}\cdot\mathbf{x},\\ (r\mathbf{x})\cdot\mathbf{y} =& r(\mathbf{x}\cdot\mathbf{y}),\\ \mathbf{x}\cdot(\mathbf{y}+\mathbf{z}) =& \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\cdot\mathbf{z}.\end{aligned}$$

As a special case,

$$\mathbf{x}\cdot\mathbf{x} = x_1^2 + x_2^2 + \ldots + x_n^2 = |\mathbf{x}|^2,$$

which implies that

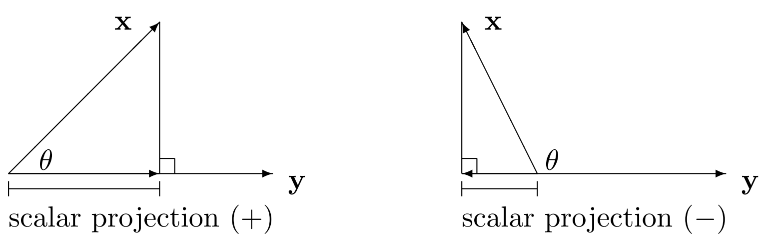
$$|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}.$$

Let θ be the angle between the n-vectors $\mathbf{x}$ and $\mathbf{y}$. Then θ satisfies

$$\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}||\mathbf{y}|\cos\theta.$$

Consequently, assuming $\mathbf{x}$ and $\mathbf{y}$ are nonzero, $\mathbf{x}\cdot\mathbf{y} = 0$ if and only if the angle between $\mathbf{x}$ and $\mathbf{y}$ is $\frac{\pi}{2}$; i.e., $\mathbf{x}$ and $\mathbf{y}$ are perpendicular. If $\mathbf{x}\cdot\mathbf{y} = 0$, then $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal**.

Let $\mathbf{x}$ and $\mathbf{y}$ be nonzero n-vectors. To find the **scalar projection** of $\mathbf{x}$ on $\mathbf{y}$, place $\mathbf{x}$ and $\mathbf{y}$ tail to tail, then drop a perpendicular line from the head of $\mathbf{x}$ to the line through $\mathbf{y}$, as shown below. The scalar projection of $\mathbf{x}$ on $\mathbf{y}$ is the directed length of the resulting line segment: positive if it lies in the same direction as $\mathbf{y}$, and negative if it lies in the direction opposite to $\mathbf{y}$. The scalar projection of $\mathbf{x}$ on $\mathbf{y}$ is also called the **component** of $\mathbf{x}$ in the direction of $\mathbf{y}$.



From the diagram, we see that the scalar projection of $\mathbf{x}$ on $\mathbf{y}$ is

$$|\mathbf{x}|\cos\theta = \frac{|\mathbf{x}||\mathbf{y}|\cos\theta}{|\mathbf{y}|} = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|} = \mathbf{x}\cdot\widehat{\mathbf{y}}.$$

The **vector projection** of $\mathbf{x}$ on $\mathbf{y}$ is the vector parallel to $\mathbf{y}$ that is constructed as shown in the diagram. It is given by

$$(\mathbf{x}\cdot\widehat{\mathbf{y}})\widehat{\mathbf{y}} = \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{y}|^2}\mathbf{y}.$$

Vectors and matrices

An n-vector $\mathbf{x}$ can also be viewed as an $n\times 1$ column matrix. Using this convention, the dot product of n-vectors $\mathbf{x}$ and $\mathbf{y}$ can be written as $\mathbf{x}^T\mathbf{y}$.

$$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \ldots + x_ny_n = \mathbf{x}\cdot\mathbf{y}.$$

An n-vector can be multiplied by an $m \times n$ matrix, and the result is an m-vector. For example,

$$\begin{pmatrix} 3 & 1 & -1 \\ 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 19 \\ 10 \end{pmatrix}.$$