University of Manitoba
Department of Mathematics

Graduate Comprehensive Exam in Algebra

10:00 AM – 4:00 PM May 7, 2018

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Instructions:

- You have altogether 6 hours to write the exam.
- Please write all your responses in the provided booklets.
- This exam has 3 parts. Part A consists of 10 questions worth two marks each. Please answer all questions in Part A.

  You have a choice of questions in each of Part B and Part C. In Part B, please answer 6 of 10 questions; each response is worth 5 marks.

  In Part C, please answer 4 of 6 questions; each response is worth 10 marks.

  If you answer more questions than required, please indicate clearly which questions you would like to be marked. Otherwise, the first 6 (resp. 4) responses will be marked.

  There are a total of 90 marks in this exam.
- In order to pass the examination, you must obtain an overall score of at least 75% ($= \frac{67.5}{90}$) in total.
- Unless otherwise stated, rings are assumed to contain a multiplicative identity 1. In your responses, it may be helpful to keep the following in mind: unless otherwise indicated,
  - vector spaces need not be finite-dimensional;
  - groups are not necessarily finite, and not necessarily abelian;
  - fields may be finite or infinite, of arbitrary characteristic;
  - rings may not necessarily be commutative;
  - S_n denotes the group of permutations of the set $\{1, \ldots, n\}$.

Part A

Answer each of the following 10 questions; each question is worth 2 marks. Your responses should be brief, two or three sentences should suffice.

A.1. Describe all the ideals of the ring $\mathbb{Q} \oplus \mathbb{Q}$.

A.2. Define what it means for an extension of fields L/K to be *separable*.

A.3. Let K be a field. Find an ideal of $K[x, y]$ that is prime but not maximal. Be sure to justify your answer.

A.4. Let A be a $n \times n$ skew-symmetric matrix (i.e. $A^T = -A$) with real coefficients, where n is odd. Show that A is not invertible.

A.5. Let R be a ring and $I \subset R$ a non-trivial proper ideal. Prove or disprove: I is a free R-module.

A.6. Let R be a ring. Prove or disprove: the sum of two nilpotent elements in R is nilpotent.

A.7. Show that $\mathbb{Z}$ is not Artinian.

A.8. Show that the centre of a group is a normal subgroup.

A.9. Let V be a vector space. Show that any maximal linearly independent set of vectors in V is a basis.

A.10. Show that every group of order n is isomorphic to a subgroup of S_n.

Part B

Answer 6 out of the following 10 questions. Each question is worth 5 marks. If you answer more than 6, please indicate clearly which questions you would like to be marked; otherwise, the first 6 responses will be marked.

B.1. Suppose T is a self-adjoint operator on a complex Hilbert space V, and v_1 and v_2 are eigenvectors with eigenvalues $\lambda_1 \neq \lambda_2$. Prove that v_1 and v_2 are orthogonal.

B.2. Prove that S_n, the group of permutations on n elements, is generated by the set of all transpositions $\{(ij) \mid 1 \leq i < j \leq n\}$.

B.3. Recall that an abelian group G is *divisible* if for each $g \in G$ and each positive integer n, there exists $k \in G$ such that $g^n = k$. Show that any quotient of a divisible group is divisible. Show that every non-trivial divisible abelian group is infinite.

B.4. Let F be a finite field of characteristic p. Show that the Frobenius map $\varphi\colon F \to F$, defined by $\varphi(x) = x^p$, is a bijection.

B.5. Give an example of each of the following, with a brief justification:

(a) A PID (principal ideal domain) that is not Euclidean.

(b) A unique factorization domain that is not PID.

B.6. Show that a finite dimensional vector space is isomorphic to its dual space.

B.7. Classify all groups, up to isomorphism, of order 10; present your classification in terms of generators and relations.

B.8. Show that if $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is odd, then $\mathbb{Q}(\alpha^2) = \mathbb{Q}(\alpha)$.

B.9. Show that $(\mathbb{Z}/10\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z})$ is cyclic as an abelian group, and find its order.

B.10. Show that if H is a subgroup of G of index 2, then H is normal.

Part C

Answer 4 out of the following 6 questions. Each question is worth 10 marks. If you answer more than 4, please indicate clearly which questions you would like to be marked; otherwise, the first 4 responses will be marked.

C.1. Let p be a prime, and $\mathbb{F}_p$ the finite field with p elements. Show that for each integer $n > 1$ there exists an extension $F_n/\mathbb{F}_p$ of degree n, and that F_n is unique up to isomorphism.

Hint: Consider the polynomial $x^{p^n} - x \in \mathbb{F}_p[x]$.

C.2. (a) Give the definition of a *solvable group*.

(b) Suppose G is a group and $N \subset G$ a normal subgroup. Show that G is solvable if and only if both N and G/N are solvable.

C.3. Consider the field $K = \mathbb{Q}(\sqrt[4]{3}, i)$. Show that K is Galois over $\mathbb{Q}$ and find its Galois group.

C.4. Let p be a prime. Show that any group of order p^2 is abelian.

C.5. Let R be a commutative ring with unity and M an R-module. Show that M is simple if and only if it is isomorphic to a module of the form R/I for some maximal ideal $I \subset R$.

C.6. Let R be a ring, and M an R-module. Show that the (contravariant) functor $\mathrm{Hom}_R(\cdot, M)$ is right exact, i.e. given an exact sequence

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of $R-$modules, show that the induced sequence

$$0 \longrightarrow Hom_R(C, M) \longrightarrow Hom_R(B, M) \longrightarrow Hom_R(A, M),$$

is exact.