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Hamiltonian Graphs

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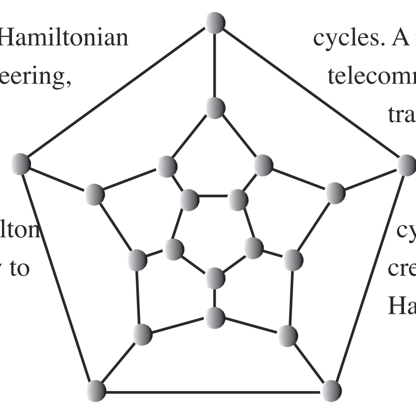
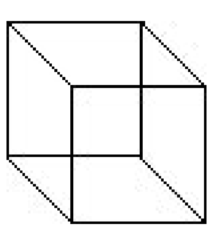
Possible Topics for Science Fair

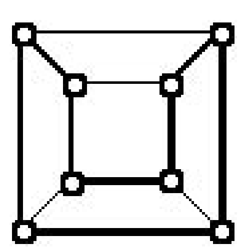
Whenever we think of Science Museums, we usually imagine huge skeletons of dinosaurs, dancing of colourful laser beams, holograph paintings, computer animations, photos from space etc., but never take a random walk through prime numbers! We do not see too many math displays simply because it is thought of as an 'abstract' science of ideas and not as concrete as, say, physics or engineering. But the fact is that most of the abstract mathematical concepts do arise from very concrete physical and biological situations and hence it is certainly possible to design dynamic and interactive mathematical exhibits. Here, by 'interactive' we do not necessarily mean just pushing buttons but the ability of the exhibit to take along the mind of the audience to a conceptual tour of the area. The point is that even though mathematics is abstract, interest in mathematics can be stimulated by exhibits that are beautiful, mind-teasing, or practical or both. The purpose of this section is to offer some examples of famous problems in mathematics which are ideal for designing math displays at the high school level.

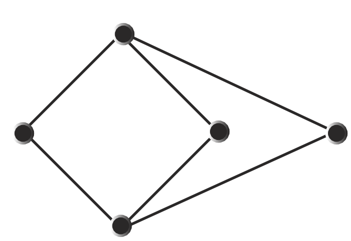
In this issue, we suggest the problem of Hamiltonian cycles. A network of points, or vertices, and lines, or edges, is called a graph. In many areas of engineering, telecommunications and computer science, researchers often need to know if there is a path that travels along the edges of a given graph visiting each vertex once, and only once, and returning to the starting vertex. Such a path is called an Hamiltonian cycle. To find whether such a path exists, or not, for a given graph is the celebrated Problem of Hamilton cycles. This is a hard problem, in a very technical sense. However, it is relatively very easy to create graphs having Hamilton cycles and otherwise. See the examples given below where Hamilton cycles are given by dark edges.



So here is the actual project. Prepare several sets of graphs: some having Hamiltonian cycles (for example, the graph above) and others not having such a cycle. Consult the websites given on Page 5, to find some good non-trivial examples of such graphs. Make transparencies of the graphs having Hamiltonian cycles and mark the cycle in colour with bold edges as shown in our examples. Ask the audience to find Hamiltonian cycles in the graphs in your poster. Then you can overlay your transparencies to reveal the actual Hamiltonian paths. There are examples of almost similar graphs, one having such a cycle and the other none! For example, the planar graphs corresponding to all the five Platonic solids are Hamiltonian. Consult the websites for examples of either kind and properties which guarantee the existences of such Hamiltonian cycles. Good luck in your project.







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MANITOBA MATH LINKS WEB SITE
........www.umanitoba.ca/science/mathematics
(a PDF Version of Math Links is available on this website)

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MANITOBA MATH LINKS

The Math Links Newsletter is published by the Mathematics Department Outreach Committee three times a year (Fall, Winter & Spring).

CAREER PROFILES IN MATHEMATICS

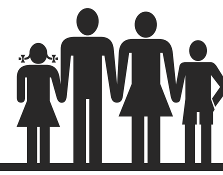
The following web sites contain short descriptions of jobs held by individuals with a mathematics degree and a wide variety of backgrounds:
http://www.siam.org/careers/career2.htm#numberfive
http://www.maa.org/careers/index.html

A NOTE FROM THE EDITORS:

We welcome comments from our readers and value their advice. Do you have any suggestions for improving Math Links? Topics for new articles? Drop us a line....either by E-mail or regular mail... to the attention of our Managing Editor. We enjoy hearing from you...!

IMPORTANT DATES TO REMEMBER:

Information Days @ U Of M
February 17th & 18th



SPEAKERS AVAILABLE

If you are interested in having a faculty member come to your school and speak to students, please contact our Managing Editor at *kangass@cc.umanitoba.ca* or phone 474-8703.

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Statistics Is For You, For Me, and For Everyone

Smiley Cheng & Brian Macpherson
Statistics Department

Aspirin can reduce risk of heart attack! There is a 30% chance of rain today! The chance of winning this special fundraising lottery is 1 in 250! Carlos Degaldo's batting average for the 2003 season was .302! Khari Jones' passing percentage for the season is 58%!

Headlines such as those above are ones we frequently see, read, or hear in the news media and they are actually examples of what can be referred to as "Statistics"!

Statistics are everywhere and are used by almost everybody, everyday. Most people however, are not fully aware that when they read or hear such items they are consuming statistical information. Moreover, statistics are widely used and applied in a vast array of disciplines such as agriculture, medicine, engineering, business, science, social science, humanity science and arts.

Statistics as a discipline is not simply a branch of mathematics, although it does make extensive use of mathematical techniques as fundamental and important tools. The two crucial ingredients to enable you to learn and study statistics are quite simply - common sense and a logical mind!

What do statisticians do? Consider the following statement made by a famous person - Florence Nightingale:

STATISTICS

> *The most important science in the whole world; for upon it depends the practical application of every other science and of every art;*
>
> *the one science essential to all political and social administration, all education, all organization based on experience, for it only gives results of our experience.*
>
> *Florence Nightingale*
> *Statistician*

The world is filled with uncertainty. Yet we are continually faced with situations where we have to make decisions without full knowledge of the background or of how our decision may impact our future. Of course we act carefully, we try to think logically while determining our priorities and assessing the risks and rewards of our decision. Using the mathematical theory of probability, statisticians/statistical scientists have formalized this decision-making process in order to understand and improve it.

Decision making in the face of uncertainty involves collecting appropriate information, evaluating it, and drawing conclusions. The information might be a measure of the sweetness of a test group's favorite blend of fruit juice, the reoccurrence rate of breast cancer in a group of women under treatment, or the velocity of a burning gas eruption on the Sun's surface.

Statisticians frequently provide crucial guidance to researchers in refining their research objectives, in determining what information should be collected, in assessing the appropriateness of the data gathered, and in measuring the reliability of predictions made from the collected information. They provide assistance in the search for clues to the solution of a scientific mystery, and very often keep investigators from being misled by false impressions. Statisticians are now viewed as being necessary and crucial partners in research projects in virtually all fields of study.

Statistical methods are usually developed in a particular context, but then find use in a range of endeavors. For example, experimental techniques that help farmers choose appropriate varieties of wheat, also assist manufacturers in improving their products, and are a key part of the testing of therapeutic drugs before they are approved for the general public. Similarly, methods used to study radio waves from distant galaxies, also help to analyze hormone levels in the blood, fluctuations in financial markets and concentrations of atmospheric pollutants. In each of these cases, statistical principles designed to solve one problem have proven to be helpful in solving other problems in very different disciplines. This diversity of application is an exciting aspect of the field, and is one reason for the continuing strong demand for well-trained statisticians.

Let us quote from an article "*Looking for work? Try a career in NBA"* from "The Saskatoon Star Phoenix" of March 18, 1999.

"Professional baseball player, president of the United States and jeans-wearing cowboy - all great jobs, right? Wrong, says a new book."

The article describes some findings that were published in a book Jobs Rated Almanac by Les Krantz (St. Martin's Griffin, 330 pages). Krantz used statistics from the Department of Labor, the U.S. census and telephone surveys to rank 250 jobs according to criteria of income, stress, physical demands, potential growth, job security and work environment. The study found that "...nine of the top 10 jobs were in computer or math-related fields, with Web site managers at the top of the heap. The worst were manual labour jobs in traditionally troubled fields, such as fishermen (No.248), lumberjacks (No. 249) and oil field roustabouts (No. 250)." According to the criterion of "Best Working Environment" the study showed the ranking of the five best careers to be:

No. 1 - Statistician
No. 2 - Mathematician
No. 3 - Computer systems analyst
No. 4 - Hospital Administrator
No. 5 - Historian

Statisticians are continually faced with interesting and varied problems to tackle. Each day brings with it exciting new challenges and satisfying rewards. Good statisticians are never bored!

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A-Mazeing Mathematics: Find Your Way Through Any Maze- Guaranteed

P. W. Aitchison
Mathematics Department

In the last issue we asked you to think of ways to find a path through a maze without getting lost or confused. Picture yourself at the start of a European hedge maze, or a Manitoba corn maze with a large number of passageways between hedges or corn plants, connected in a complicated arrangement. The objective is to find a route through the maze to a specified destination. Soon after you start you will encounter junctions with many different route choices but you do not have a map or any way to distinguish the correct route from unproductive routes.

Surprisingly, you can traverse the maze successfully using just a few simple rules, no matter how large or complicated the maze. The only tool you need is a method to mark the entrances and exits from each junction you encounter; a number of identical pebbles that you can leave as junction markers. You can pick up all of the pebbles when you exit, leaving no sign that you had ever been there.

The rules are just a list telling you exactly what to do in every possible situation in the maze. Mathematicians and Computer Scientists call this list an ***algorithm***. The one given here is a variation of one of the best known algorithms of graph theory, ***depth-first search***, used for searching in graphs (that is, graphs that are a collection of nodes connected by edges). You use the algorithm on a maze by applying step 1 at the entrance to the maze, then every time you encounter a junction or a dead-end you choose the appropriate action from the list.

The algorithm (list of rules):

1) ***First time at a junction and forward travel:*** Put *two pebbles* at the passageway first used to enter a junction (including the entry to the maze). Choose any other passageway, put *one pebble* at the start and follow the passageway. This type of exploration is called *forward travel*.

2) ***In forward travel if you encounter a dead end:*** Simply reverse direction, but now you are in *backward travel* mode (often called backtracking).

3) ***In forward travel if you encounter a junction previously visited:*** That is, the junction already has pebble markers. *Do not enter this junction*. Leave one marker pebble and reverse direction. You are now in *backward travel*.

4) ***In backward travel if you come to a junction:*** This is a previously-visited junction because you have previously been in *forward travel* along this passageway. There are two possibilities:

a) *At the junction, there is a passageway without a pebble marker:* Put a pebble marker at this passageway and follow it. You are now back in *forward travel again*.

b) *At the junction, all passageways have pebble markers*: Pick up all of the pebbles at the junction (it will never be visited again) and exit through the single passageway that had two pebble markers. You are still in *backward travel* mode.

That's all of the exploration rules! However, we have left out a couple of crucial points. How does the algorithm stop?

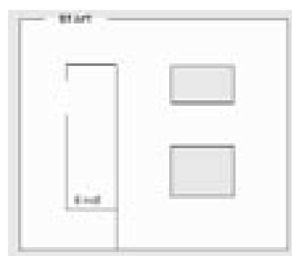
Stopping the algorithm: The algorithm ends naturally in two possible ways, both of which would be self-evident if you were traversing a real maze:

- ***Success***: In *forward travel* mode you are successful and reach the end (destination) which is presumably some recognizable place. At this point there is a natural way to exit the maze. You reverse direction so you are in *backward travel* and follow rule 4b at junctions, without exploring anymore untraveled passageways. This will take you back to the start of the maze collecting pebbles on the way back.
- ***Failure:*** In *backward travel* mode you are back at the original entrance to the maze and there are no more untraveled passageways there, which means that all parts of the maze have been explored. In this case rule, 4b applies and you will simply exit the maze. This can only happen if there is no path to the destination. Actually this could also be a successful conclusion if your intention was not to reach a particular destination, but to simply explore all possible passageways in the maze.

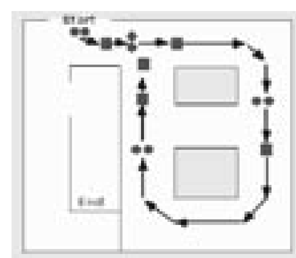
An example:

Here is a simple example; the same process works for a maze of any size or complexity. The search uses the arbitrary rule that where there is a choice between two passageways, then always choose the left-hand passageway. The symbols used are:

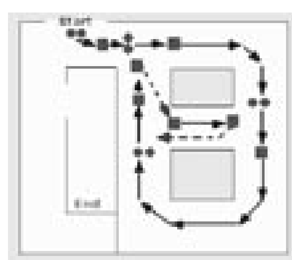
- Two circular dots represent the two pebbles used to mark the first passageway into a junction.
- One square marker represents a single pebble at every passageway used to leave a junction.
- Solid arrow represents forward travel.
- A dashed arrow represents backward travel.



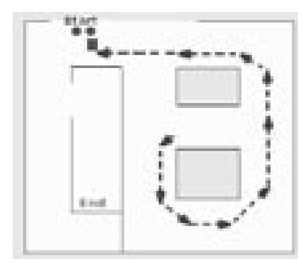
The Maze



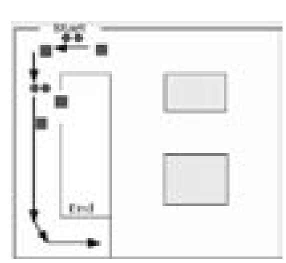
Forward:placing pebbles at junctions



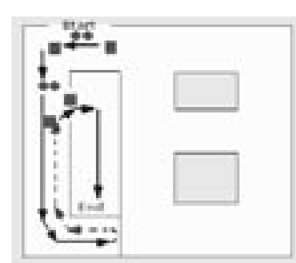
Backward travel then forward travel on new section, then backward travel again



Backward: Retrieve pebbles



Forward travel to a dead end



Back then forward

Conclusion: The maze algorithm can be proved to work for any maze, even a 3-dimensional maze that cannot be drawn on a piece of paper. But algorithms can work, yet still be of little use. A famous example of this is the *Travelling Salesman Problem* that asks you to find the route with the least possible travel time for a salesperson visiting a large number of cities. This apparently simple problem does have a very simple solution algorithm, but it is not much use because it takes far too long to process the algorithm. Even if there are only 50 cities the solution can be absurdly time consuming even on very powerful computers.

The maze algorithm is very efficient because you never travel along a passageway more than twice. Hence the time for carrying out the algorithm is just a linear relationship with the length of passageways. In algorithmic terms, a linear algorithm like this is considered to be very good. The so-called worst case situation is a maze where every passageway is travelled exactly twice before finding the destination. For example a maze has exactly two passageways at the entrance and one goes immediately to the destination. If you chose the other passageway first then you will traverse every passageway twice, eventually returning to the entrance where you will immediately find the destination after going into the second passageway.

Is this the most efficient algorithm possible? The answer depends on how you define "efficient algorithm". If it means less total length of passageways travelled before finding the destination, then the answer to the question is: *no*, it is not the most efficient. For example, on the third picture the route reversed direction into backward travel from the junction on the right hand side of the maze. However, if we had kept track of the fact that there were no further untraveled passageways on the route then we could have simply headed straight back from the far right junction towards the entrance, omitting the backward travel loop shown in the bottom right of the fourth picture. However, that version of the algorithm would leave behind uncollected pebbles in the maze and increasing the efficiency has the downside of making the algorithm more complicated.

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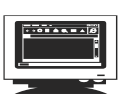

Websites for Hamiltonian Graphs

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Mathematics Department

http://www-math.cudenver.edu/~wcherowi/courses/m4408/gtln12.html

http://www.swif.uniba.it/lei/foldop/foldoc.cgi?Hamiltonian+path

http://www.nist.gov/dads/HTML/hamiltonianCycle.html

http://planetmath.org/encyclopedia/HamiltonianPath.html

http://mathworld.wolfram.com/HamiltonianGraph.html

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Definitions With e's

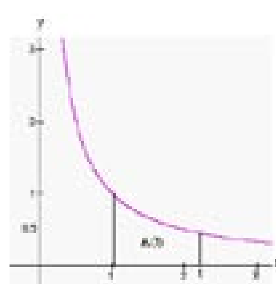
David S. Gunderson
Mathematics Department

A few special numbers occur so often in mathematics that they are given special names. For example, $\pi \approx 3.1416$ is used for the ratio of a circle's circumference to its diameter. Another famous number is the *golden ratio,* often denoted by τ or ϕ, which has a value of $(1+\sqrt{5})/2 \approx 1.61803$. (If a line segment AB is cut by a point C so that the ratios $\left|\frac{AC}{CB}\right|$ and $\left|\frac{AB}{AC}\right|$ are the same, this ratio is called the *golden ratio*. It can be found by solving $a^2 - a - 1 = 0$.)

Perhaps the next most famous real number is the constant e (to 50 decimal places):
$e = 2.71828182845904523536028747135266249775724709369995...$

It does not have just one simple definition like π, nor does it have a nice algebraic definition like τ . In high school, I was taught one definition, another in first year calculus, and finally another in integral calculus; only then did I see that all of these definitions really determined the same number (though I won't prove this fact here). Let's look at a few different ways to define e, beginning with the earliest definition to appear in print.

Using area under a hyperbola: This definition actually arises in integral calculus, but is the easiest to visualize. Examine the graph of $y = 1/x$ for $x > 0$; the graph's shape is called a hyperbola. For any real number $t \geq 1$, consider the area under this hyperbola between $x = 1$ and $x = t$ (and above the x-axis); call this area $A(t)$.



Observe that $A(1) = 0$ (since when $t = 1$, the area $A(1)$ is the area of a small vertical segment, which is zero). As t increases from 1, the area $A(t)$ grows, and with some careful graphing, one can see that A(t) eventually gets as large as one wants. Define e as that unique number (around 2.7) so that $A(e) = 1$. For those who have studied integral calculus, the more common notation for $A(t)$ is $\ln(t) = \int_1^t \frac{1}{x}\,dx$; the function "ln" is now called the "natural logarithm" .

By slopes of exponential functions: Another way to define e is in terms of exponential functions. Some examples of simple exponential functions are given by 2^x or 10^x. In these cases, the number 2 or 10 is called the *base* and x the exponent. Generally, an exponential function is given by $f(x) = b^x$ for some base b. When $b > 1$, these functions grow very fast, and their graphs are climbing when they cross the y-axis (when $x = 0$). If you graph $y = 2^x$ and $y = 10^x$ on the same axis, you will see that the second graph is 'steeper' when crossing the y-axis.

Many first year calculus books define e to be the unique base b so that the slope of the line tangent to the graph of $y = b^x$ as it crosses the y-axis is equal to 1 (which is totally uninspiring to me!). They use this as a definition, since they must first discuss slopes (derivatives) before they discuss areas (integrals). For those having seen some calculus, this definition says that e is chosen so that if $f(x) = e^x$, then $f'(0) = 1$. Some books use the notation "exp(x)" before they use e^x, to remind us of exponentiation, and so many think that this is where the "e" comes from. One can then later prove that these two definitions for e (one from ln, and one from slopes) define the same number.

(continued on Page 6)

Those who have studied logarithms know that taking logarithms and exponentiation are inverse operations; $\log_b(x) = y$ is the same as $x = b^y$. Above, we defined e in terms of area $\ln(t)$; it can be shown that in fact $\ln(t) = \log_e(t)$, the logarithm to the base e, the "natural" logarithm. The two statements $\ln(x) = y$ and $x = e^y$ have the same meaning.

By a limit arising from slopes: Some texts define e by

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n = \lim_{h\to 0^+} (1 + h)^{1/h} .$$

With $n = 2$, the expression in the limit is $(\frac{3}{2})^2 = \frac{9}{4} = 2.25$, and with $n = 3$, one gets $\frac{64}{27} \approx 2.37$, only a little closer to e. This definition is often introduced first just so that the derivative (slope) of the function $f(x) = e^x$ at $x = 0$ works out to be 1.

Newton's formula for e: In 1669 Newton published a way to compute e by an infinite sum (but he never called it e). This method is actually a consequence of some deep results in calculus but is very easy to use. In high school I was taught that

$$e = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}...$$

(recall that $0! = 1$ and $n! = 1 \cdot 2 \cdot 3 \cdot \cdots n$) What does it mean to add up infinitely many things? You add them up from the beginning, keeping a running total (called a *partial sum*), and if these running totals approach a single number, then we say that this sum *converges* to this number. We won't prove it, however you might persuade yourself that this sum converges by adding the first few terms; for example,

$$1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}= 2.5+0.166666... +.04166666... = 2.708333....$$

and continuing. Using Newton's formula, you can compute e to as many decimal places as you like fairly quickly! (Over a billion digits have been computed - see the first website listed below for the current record.) Interestingly enough, for any x, one can compute e^x in a similar method by

$$e^x = \frac{1}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+ ....$$

By continued fractions: Every real number has a so-called *continued fraction* representation; I leave it as an exercise to ponder its meaning, but the continued fraction for e is rather beautiful and so I thought that I might share it with you:

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{\ddots}}}}}$$

This can also be written as: $e = 2+\frac{1}{2}(1+\frac{1}{3}(1+\frac{1}{4}(1+\frac{1}{5}(1+...))))$.

It is hard to say who first identified this constant, though its existence was implied by the work of Napier in 1614 while studying logarithms and bases. The Swiss-German mathematician Leonhard Euler (pronounced "oiler") used the symbol e in 1727 (in a note *Meditation upon experiments made recently on the firing of a cannon*, which wasn't published until 1862), and the symbol e first appeared in a published work in 1736 (in Euler's *Mechanica*). Did Euler name the constant after himself? Probably not — my best guess is that since Euler defined (like in the first definition above) e as "that number whose hyperbolic logarithm is equal to 1". In German, "einheit" means *one-ness*, or *unity*, so it is possible that this is how the symbol e was chosen. (In modern general algebraic settings, the symbol e is often used to denote an identity element.)

None of the ways we have defined e is very simple. Euler proved e is *irrational* (not a simple fraction). In 1873, Hermite proved that e is *transcendental*, that is, is not a root of any polynomial (like the golden ratio is). The number e is often introduced in calculus, but it appears in most every branch of mathematics. Two common occurrences (see [1] for details) involve compounded interest and probability (e.g., the hatcheck problem in combinatorics). See also [3] for a few more surprising places e shows up. For me, the most surprising place is in the formula $e^{\pi i} = -1$; can you make sense of this?

Here are a few of the many wonderful websites regarding e (the last one contains "Top $\ln(e^{10})$ reasons why e is better than pi."):
http://pi.lacim.uqam.ca/eng/records_en.html
http://mathforum.org/dr.math/faq/faq.e.html
http://mathforum.org/library/topics/about_e/
http://mathworld.wolfram.com/e.html (good bibliography)
http://members.aol.com/jeff570/constants.html
http://www.mu.org/~doug/exp/
http://www.maa.org/mathland/mathtrek_11_9_98.html

References

[1] M. Gardner, "The Transcendental Number e", in *The Unexpected Hanging and Other Mathematical Diversions*, Chicago University Press, Chicago, IL, 1991.
[2] E. Maor, *e: The Story of a Number*, Princeton University Press, Princeton NJ, 1994.
[3] H. S. Shultz and B. Leonard, Unexpected Occurrences of the Number *e, Mathematics Magazin*e, Vol. 62, No. 4, October 1989

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Magic With Magic Squares

R. Craigen
Mathematics Department

Let us consider square arrays of numbers, such as S, below:

$$\mathbf{S} = \begin{array}{|ccc|} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

The numbers in a square are called its *entries*. The horizontal lines of numbers are its *rows* and the vertical lines are its *columns*. Its *diagonals* are the two lines of numbers running corner-to-corner: its *forward diagonal* (top left to bottom right) and *back diagonal* (top right to bottom left). Its *row sums* are the sums of the entries in the rows of the square; *column sums* and *diagonal sums* are defined similarly. Collectively the rows, columns and diagonals are called the *lines* of the square. The *order* of a square is the number of its rows.

Thus, the square S above has order 3; its entries are 1,2,...,9. Its first row is (1,2,3) and its third column is (3,6,9), its two diagonals are (1,5,9) and (3,5,7). Its three row sums, three column sums, and two diagonal sums are 6, 15, 24; 12, 15, 18; 15 and 15, respectively.

Can the numbers in S be rearranged so that the eight line sums are all equal? More generally, can the numbers $1,2,...,n^2$ be arranged in a square, all of whose line sums are equal? Such a square is called a *magic square*.

Magic squares have fascinated people for millenia. The ancient Chinese called them "Lo-Shu", and they have appeared in old arabic texts and in Medieval art. Serious mathematicians study magic squares, but so do many hobbyists. Many famous people, such as Benjamin Franklin, have been interested in them. The medieval artist Albrecht Dürer's engraving, *Melancholia*, includes a magic square of order 4 that cleverly reveals the year in which it was made, 1514. Some have attributed mystical properties to magic squares, as their name suggests. Others just find them an interesting puzzle.

Our array S above is clearly not magic---while four of the eight line sums have the same value (15) the other four are different. There is no magic square of order 2 (try it and see why!). Rearranging the 9 numbers in S randomly will not easily produce a magic square. If you try, you may decide that the task is impossible. Instead, let us tackle the problem analytically.

Call the common line sum, s, for a magic square of order n its *magic sum*. What would be the magic sum for a square of order 3? Since the three row sums are s, and every entry appears in exactly one row, the sum of the entries of the square, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, must be equal to $3s$. The magic sum is therefore $s = 45/3 = 15$. There are exactly 8 ways to get 15 as a sum of three distinct numbers from $\{1,2,3,...,9\}$: $15 = 1 + 5 + 9 = 1 + 6 + 8 = 2 + 4 + 9 = 2 + 5 + 8 = 2 + 6 + 7 = 3 + 4 + 8 = 3 + 5 + 7 = 4 + 5 + 6$. Since an order three square has 8 line sums, each of the above sums must occur.

The middle entry of the square is in four lines (a row, a column, and both diagonals), and 5 is the only number appearing in four of the above sums, so the middle entry must be 5. Similarly, since each of the four corners of the square appears in three lines they must be 2, 4, 6 and 8. Since the diagonals sum to $s = 15$, the pairs (2, 8) and (4, 6) appear in opposite corners. Thus, 2 and 4 appear along a side, and the number between them must be $15 - 2 - 4 = 9$, and this side is (2, 9, 4). Similarly the other sides are (4, 3, 8), (8, 1, 6) and (6, 7, 2); with this information you can now construct the square , try it!

Every now and then some enthusiast with more time to play than to read claims to have constructed "the world's largest known magic square". I will demonstrate, momentarily, why this is a silly claim, akin to claiming to have discovered "the largest positive integer"—anyone who knows a simple trick can always produce a larger one.

Let us calculate the magic sum, s, for a square of order n. The sum of the entries must be $1 + 2 + \cdots + n^2 = n^2(n^2 + 1)/2$ (can you see how to get this?). Since each number from 1 to n^2 appears in exactly one row and all n rows have sum s, we must have $ns = n^2(n^2 + 1)/2$. Thus, $s = n(n^2+1)/2$. Taking $n = 3$, we obtain $s = 3(3^2 + 1)/2 = 15$, as before. The magic sum for $n = 4$ is $s = 4 \cdot 17/2 = 34$.

Do you suspect that our derivation of order 3 squares above can be used to get larger squares? If so, try it in order 4. You will not find the analysis very helpful because there are too many possibilities; in larger orders, the situation is even worse.

But here is a way to find an order 3 square that *does* generalize for larger squares. We begin by forming two (non-magic) squares, P and Q, as follows:

The first row of P is (3, 0, 6) and the first row of Q is (1, 3, 2). Subsequent rows of P are obtained by *circulating* the entries of the previous row to the right—by this we mean that each entry is shifted to the right by one position, except for the last entry, which we move to the first position. Similarly we obtain subsequent rows of Q by circulating the first row to the left:

$$P = \begin{array}{|ccc|}\hline 3 & 0 & 6 \\ 6 & 3 & 0 \\ 0 & 6 & 3 \\ \hline\end{array} \qquad Q = \begin{array}{|ccc|}\hline 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ \hline\end{array}$$

Add P and Q entry by entry, and a magic square "magically" appears:

$$P + Q = \begin{array}{|ccc|}\hline 3+1 & 0+3 & 6+2 \\ 6+3 & 3+2 & 0+1 \\ 0+2 & 6+1 & 3+3 \\ \hline\end{array} = \begin{array}{|ccc|}\hline 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \\ \hline\end{array}$$

How were the first rows of P and Q chosen?
Observe that every number from 1 to 9 can be expressed in exactly one way as a sum of an element of $A = \{0, 3, 6\}$ and one from $B =\{1, 2, 3\}$, and that every such sum appears in $P + Q$. Every element of A appears in each line of P except the forward diagonal, so the corresponding line sums will all be $0 + 3 + 6 = 9$. Similarly, every line of Q, except for the back diagonal, will have sum $1 + 2 + 3 = 6$. So every row and column sum of $P + Q$ will be $9 + 6 = 15$.

Now let us account for the diagonal sums. As observed above, the forward diagonal sum of Q is 6. In order for the forward diagonal of $P + Q$ to have the magic sum 15, the forward diagonal of P must be 9. Since this diagonal is constant, the first row of P must begin with 3. Similarly considering the back diagonal, we see that the first row of Q must end with 2. We thus ensure that the diagonal sums of $P + Q$ are 15, and the square is magic.

Let us try this for order 5: The magic sum is $5(5^2 + 1)/2 = 65$. Every number from 1 to 25 appears exactly once as a sum of an element of $A = \{0, 5, 10, 15, 20\}$ and an element of $B = \{1, 2, 3, 4, 5\}$. If as above, we form squares P (circulate a row composed of the elements of A to the right) and Q (circulate a row composed of the elements of B to the left), $P + Q$ will have constant row and column sums equal to $0 + 5 + 10 + 15 + 20 + 1 + 2 + 3 + 4 + 5 = 65$.

The forward diagonal sum of Q will be $1 + 2 + 3 + 4 + 5 = 15$, so the first entry of the first row of P must be $(65 - 15)/5 = 10$. Similarly we determine that last entry of the first row of Q must be $(65 - 50)/5 = 3$. We now have enough information to form P and Q, ensuring only that these conditions are satisfied. Their sum is $P + Q =$

$$\begin{array}{|ccccc|}\hline 10 & 0 & 5 & 15 & 20 \\ 20 & 10 & 0 & 5 & 15 \\ 15 & 20 & 10 & 0 & 5 \\ 5 & 15 & 20 & 10 & 0 \\ 0 & 5 & 15 & 20 & 10 \\ \hline\end{array} + \begin{array}{|ccccc|}\hline 1 & 2 & 4 & 5 & 3 \\ 2 & 4 & 5 & 3 & 1 \\ 4 & 5 & 3 & 1 & 2 \\ 5 & 3 & 1 & 2 & 4 \\ 3 & 1 & 2 & 4 & 5 \\ \hline\end{array}$$

$$= \begin{array}{|ccccc|}\hline 11 & 2 & 9 & 20 & 23 \\ 22 & 14 & 5 & 8 & 16 \\ 19 & 25 & 13 & 1 & 7 \\ 10 & 18 & 21 & 12 & 4 \\ 3 & 6 & 17 & 24 & 15 \\ \hline\end{array}$$

Check that this is indeed a magic square.

This method can be used to produce squares of arbitrarily large *odd* order. See if you understand it well enough to:

a) construct a square of order 7;

b) construct a couple of squares of order 5 different from the one above.

The next issue of Math Links will contain another article on Magic Squares, and a series of exercises based on the construction described here.

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Problem Corner

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Dear Readers:

Welcome once again to PROBLEM CORNER. Here is the problem from the last column and its solution.

Problem:
If $n + 1$ integers are chosen from the first $2n$ positive integers {1,2,...,2n}, at least one of them must divide another.

Solution:
By factoring out 2's, every positive integer can be expressed in the form $a2^b$, where a is an odd integer and b is a nonnegative integer. For example, $24 = 3 \cdot 2^3$, $64 = 1 \cdot 2^6$, and $25 = 25 \cdot 2^0$. In the set of integers $\{1,2,...,2n\}$, n are odd and all are different. These will be possible values for a. When 2's are factored from the even integers, values for a will be repetitions of those for the odds. In other words, there are only n values for a. Since $n + 1$ integers are being chosen, the Pigeon Hole principle states that at least two of them must have the same value for a. Let them be $a2^b$ and $a2^c$. Clearly one of these divides the other.

Here is your new problem:
A psychologist observes groups of four individuals at a time. In how many ways can the psychologist choose 5 groups of 4 from among 20 people?

Send submissions on this problem to:

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At a press conference held at the White House, President George W. Bush accused mathematicians and computer scientists in the U.S. of misusing classroom authority to promote a Democratic agenda. "Every math or computer science department offers an introduction to AlGore-ithms," the president complained. "But not a single one teaches GeorgeBush-ithms..."

Bargain Math Books

David S. Gunderson
Mathematics Department

The University of Manitoba Bookstore has recently stocked a large number of Dover paperbacks in mathematics; these include reprints of many classics, and most are a very affordable price. The Bookstore also has a wide selection of other inexpensive math books. In this column, we review both classics and new releases, but will concentrate on those that are bargains.

Heinrich Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solutions,* translated by David Antin, Dover Publications, New York, 1965. Price: $21.95. Paperback, 395 pages.

What is remarkable about this book is the selection of problems. Most look familiar; here they are given names and extensive references. Dörrie has addressed head-on a significant number of fundamental questions in mathematics. Most problems are understandable by an energetic high school student. Many of the solutions are highly non-trivial, yet revealing the power of elementary techniques. Some of the discussions are a bit difficult to follow, probably because this work was originally in German and the translation might be too literal in spots. Here are some of the topics: the transcendence of e, the fundamental theorem of algebra, the fact that every prime number of the form $4n + 1$ can be expressed as a sum of two squares in precisely one way, trisecting an angle, and Newton's exponential series. This book can be a fantastic resource for teachers and energetic students alike. I only discovered this book a year ago, and now wonder how I survived so long without it!

Mario Livio, *The Golden Ratio: The Story of Phi, The World's Most Astonishing Number,* Broadway Books, New York, 2002. Price: $22.95. Paperback, 294 pages.

The golden ratio is often denoted by τ (tau) or ϕ (phi, pronounced "fee") and has a value $(1 + \sqrt{5})/2 \approx 1.61803$. If a line segment AB is cut by a point C so that the ratios $\left|\frac{AC}{CB}\right|$ and $\left|\frac{AB}{AC}\right|$ are the same, this ratio is called the *golden ratio* (or sometimes the "*golden mean*"). This number is found in the most unexpected places. For example, it is the ratio between the diagonal and side lengths in a regular pentagon. It is found in many aspects of art and architecture. It also appears in various topics which have nothing to do with geometry. This book takes the reader on a very gentle stroll through different gardens of science, nature, and art, slowing down just enough to point out where phi is hidden. The history and context surrounding each is fascinating!

I have only one warning, this is a book that you will peek at while coming home from the bookstore and then get lost in it immediately. If you read it in bed, you might not want to turn the lights out. I think that any high school student will enjoy this volume; you need not be a math nut.

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