UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS

Graduate Comprehensive Exam in Topology

Monday April 30, 2018

10:00 to 16:00 (NO EXTRA TIME PERMITTED)

Examiners: L. Butler
J. Chipalkatti
D. Krepski (coordinator)

INSTRUCTIONS:

You have **six** hours to complete the exam.

The exam consists of six (6) pages, including this cover page.

Answer all eight (8) of the questions in Part A, which is worth a total of 40 marks, distributed according to the following table:

Question	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
Value	4	4	5	7	5	5	4	6

For each of Part B and Part C you have a choice of questions. Answer any three (3) of the four (4) 10 mark questions in each part. Parts B and C are worth a total of 30 marks each.

For either Part B or Part C, you may attempt all four questions in that Part; however, only three answers will be evaluated. **If you submit responses to all four questions appearing in that part, clearly indicate which responses are to be evaluated**. In the absence of any other indication, the first three responses will be evaluated according to the order in which they appear.

To pass this exam, you must obtain a score of at least 75% overall.

PART A Short answer questions

Answer all of Questions 1 through 8. The point values for each question are indicated in square brackets [ ].

[4] **Question 1.**
Let X be a non-empty set with the co-countable topology (i.e. non-empty open sets are complements of countable sets). Show that X is Lindelöf.

[4] **Question 2.**
Let X be a Hausdorff topological space. Suppose $Y \subset X$ is a compact subspace, and let $x \in X \smallsetminus Y$. Show that there exist disjoint open subsets $U, V \subset X$ containing x and Y, respectively.

[5] **Question 3.**
Let X be a non-empty set. Determine the topologies τ on X such that every function $f : (X, \tau) \to (X, \tau)$ is continuous.

[7] **Question 4.**

(a) Suppose X is a topological space equipped with an equivalence relation $\sim$. Define the quotient topology on the set of equivalence classes $X/\sim$.

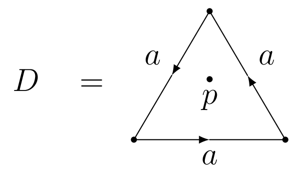
(b) Suppose we are given an equivalence relation $\sim$ on $X = \mathbb{R}^n$ and that $x_0 \in \mathbb{R}^n$ is the only point in its equivalence class. Show that $A = \{[x_0]\} \subset X/\sim$ is nowhere dense.

[5] **Question 5.**

(a) Define what it means for a continuous map $f : X \to Y$ to be a *covering map.*

(b) Let $q : \mathbb{R}^2 \smallsetminus \{(x, y) \mid xy = 0\} \to \mathbb{R}^2_+$ given by $q(u, v) = (|u|, |v|)$, where $\mathbb{R}^2_+$ denotes the first quadrant. Decide whether q is a covering map, and provide a brief justification for your response.

[5] **Question 6.**
Let $D = X/\sim$, where X denotes the triangle in the plane (including the interior) with sides identified according to the diagram below.

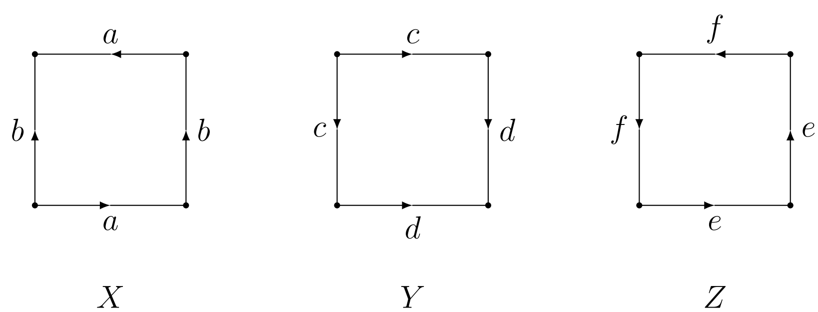


(a) Compute the fundamental group of $D \setminus \{p\}$, where p is a point in D, as shown.

(b) Compute the fundamental group of D.

[4] **Question 7.**
Let $q : S^2 \to \mathbb{R}P^2$ denote the universal covering map. Show that there does not exist a *section* of q—that is, a continuous map $\sigma : \mathbb{R}P^2 \to S^2$ satisfying $q(\sigma(x)) = x$ for all $x \in \mathbb{R}P^2$.

[6] **Question 8.**
Two of the three surfaces X, Y, Z below are homeomorphic. Decide which two surfaces are homeomorphic and use Tietze transformations (also called elementary transformations) to justify your response.



PART B Point-set Topology

Answer 3 of Questions 9 through 12:

[10] **Question 9.**
Let X be a topological space.

(a) Form true statements by selecting one of the underlined terms in each sentence.

i) If X is regular and *separable, second countable, first countable*, then X is normal.

ii) If X is locally compact and Hausdorff, then X is *regular, completely regular, normal*.

iii) If X is *separable, first countable* and metrizable, then X is second countable.

(b) Prove one of the (true) statements in (a).

[10] **Question 10.**
Let X and Y be non-empty sets and $f : X \to Y$ a function (of sets). Let $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ denote the power set of X and Y, respectively.

(a) Suppose that $\mathcal{T} \subset \mathcal{P}(Y)$ is a topology on Y. Show that $f^*(\mathcal{T}) = \{f^{-1}(V) \mid V \in \mathcal{T}\}$ is a topology on X.

(b) Suppose that $\mathcal{S} \subset \mathcal{P}(X)$ is a topology on X. Show that $f_*(\mathcal{S}) = \{U \mid f^{-1}(U) \in \mathcal{S}\}$ is a topology on Y.

(c) To compare the topologies $\mathcal{T}$ and $f_*(f^*(\mathcal{T}))$ on Y,

(i) determine which topology is always contained in the other; and

(ii) give a condition on f that guarantees the two topologies coincide.

(Be sure to provide justifications for your response.)

[10] **Question 11.**

(a) State Urysohn's Lemma.

(b) Let X be a compact Hausdorff space, and $A, B \subset X$ disjoint closed subsets. Suppose there exists a continuous map $g : X \to \mathbb{R}^n$ whose restrictions $g\big|_{X \smallsetminus A}$ and $g\big|_{X \smallsetminus B}$ are injective. Show that there exists a topological embedding (i.e. a homeomorphism onto its image) $G : X \to \mathbb{R}^{n+1}$

[10] **Question 12.**

For real numbers a and b, let $d(a,b) = \min\{|a-b|, 1\}$. Let $\mathbb{R}^\omega$ denote the countable product $\mathbb{R} \times \mathbb{R} \times \cdots$. For sequences $x = (x_1, x_2, \ldots,), y = (y_1, y_2, \ldots)$ let

$$\rho(x,y) = \sup\{d(x_n, y_n)\}_{n=1}^{\infty},$$

and let $\mathcal{T}$ be the topology defined by the metric ρ.

(a) Consider the set $V = (-1/3, 1/3) \times (-1/3, 1/3) \times \cdots$. Show that V is not open in $\mathcal{T}$.

(b) Give the definitions of the product topology and the box topology on $\mathbb{R}^\omega$.

(c) Show that the topology $\mathcal{T}$ is strictly larger than the product topology and strictly smaller than the box topology.

PART C Algebraic Topology

Answer 3 of Questions 13 through 16:

[10] **Question 13.**
Let $\Sigma = S^1 \times S^1$ denote the torus.

(a) Compute the fundamental group of $\Sigma \smallsetminus \{p\}$, where p is any point in Σ.

(b) Compute the fundamental group $\Sigma \smallsetminus \{p, q, r\}$, where p, q, and r are distinct points in Σ.

[10] **Question 14.**
Let $S^1 \subset \mathbb{C}$ denote the set of unit complex numbers. For base point preserving maps $f, g : (S^1, 1) \to (S^1, 1)$, let fg denote the pointwise product $(fg)(z) = f(z)g(z)$. (All maps below are base point preserving of S^1, and homotopies are rel $\{1\}$.)

(a) Show that if f is homotopic to f' and g is homotopic to g', then fg is homotopic to $f'g'$.

(b) Show that fg and the loop obtained by concatenation $f * g$ are homotopic. (Hint: f is homotopic to $f * e$, where e is the constant loop.)

(c) Let $f : (S^1, 1) \to (S^1, 1)$. Suppose that f^n is null homotopic for some $n > 0$. Show that f is null homotopic.

[10] **Question 15.**

(a) Let S and R be connected surfaces, and suppose $q : \widetilde{S} \to S$ is a 2-sheeted covering map. Show that there is a connected sum $S\#R$ that admits a 2-sheeted covering by $\widetilde{S}\#R\#R$.

(b) Use the fact that there exists a 2-sheeted cover $T^2 \to K$ to show that every nonorientable compact connected surface Σ_g of genus $g \geq 1$ admits a 2-sheeted cover by an orientable surface of genus $g - 1$. (Here $T^2 = S^1 \times S^1$ and $K = \Sigma_2$ denotes the Klein bottle.)

[10] **Question 16.**
Let $g, h : \mathbb{R}^2 \to \mathbb{R}$ be continuous functions such that $\| g \| + \| h \| < 1$ where $\| \bullet \|$ is the sup-norm. Prove that the equations

$$2x = y + g(x, y), \qquad 2y = x + h(x, y)$$

have a solution (x_0, y_0) such that $|x_0| + |y_0| < 1$.

END OF EXAM