University of Manitoba
Department of Mathematics

Ph.D. Comprehensive Examinations
REGULATIONS and SYLLABI

Date: June 14, 2018

Ph.D. Comprehensive Examinations

A. Regulations

1. From the Department of Mathematics Supplemental Regulations for the Ph.D. Program: "The candidacy examination in the Ph.D. program in Mathematics consists of **two** separate written comprehensive examinations chosen from the following areas: a) Algebra; b) Analysis; c) Combinatorics; d) Computational Mathematics; e) Differential Equations; f) Topology. Regulations governing these examinations and the latest syllabi on which these examinations are based are described in the document "Ph.D. Comprehensive Examinations REGULATIONS and SYLLABI" approved by the Department Council, and are available from the Associate Head (Graduate Studies). At least one of the examinations must be in Algebra or Analysis. The Graduate Studies Committee arranges comprehensive examinations three times a year, normally in January, April and September. The student must register to write a comprehensive examination by sending a request to the Associate Head (Graduate Studies) by February 1 if the examination is given in April, by July 1 if the examination is given in September and by November 1 if the examination is given in January of the following year. The choice of areas must be approved by the student's advisor.

   The standard of pass shall be given on the question sheet of each examination. A maximum of two attempts on each comprehensive examination is allowed. A student who fails a comprehensive examination in any area twice shall be required to withdraw from the Ph.D. program in Mathematics."

2. Once a student has made a formal request for an examination (either in writing or by email), he/she is obligated to write it (except that a request can be withdrawn before March 1 if the examination is given in April, before August 1 if the examination is given in September and before Dec 1 if the examination is given in January of the following year). Absence from the examination on medical or compassionate grounds will be excused according to the same policies that apply to final examinations in the Faculty of Science. Any other delay or deferral of the examination shall be considered by the Graduate Studies Committee only upon receipt of a written request from a student's advisor (this request must be received by the Graduate Studies Committee before the examination) outlining the specific reasons for the delay or deferral.

3. Each Examining Committee shall consist of at least three persons. One member of each committee will be designated as Coordinator and will be responsible for communicating with the students, with the Graduate Studies Committee through the Associate Head (Graduate Studies) and with the Department office (through the Administrative Assistant).

   The Examining Committee sets the questions for the examination as it sees fit. Once the examination has reached its final form it is the responsibility of every committee member to read the entire examination and ensure that it is consistent with the syllabus and with the general level of difficulty expected.

   The Coordinator is responsible for ensuring that all the regulations in this document are adhered to.

4. **This regulation applies to all comprehensive examinations unless the syllabus for an examination explicitly overrides it.**

   Style of the examination:

   The examination shall consist of up to three parts A, B and C.

a) Part A shall consist of N_1 mandatory questions, worth a total of W_1 marks (individual marks for each question may or may not be the same). Here, N_1 is a positive integer and W_1 is positive.

b) Part B (if any) shall provide a choice of N_2 questions worth ω_2 marks each with the instruction that the student is allowed to attempt M_2 out of N_2 questions (M_2 shall be strictly less than N_2) with a clear indication of which M_2 questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_2 := \omega_2 M_2$).

c) Part C (if any) shall provide a choice of N_3 questions worth ω_3 marks each with the instruction that the student is allowed to attempt M_3 out of N_3 questions (M_3 shall be strictly less than N_3) with a clear indication of which M_3 questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_3 := \omega_3 M_3$).

It is up to the Examining Committee to decide how many questions will be in each part and how many marks will each part be worth (*i.e.,* there are no restrictions on the values of N_i, M_i and W_i), and there is no obligation to follow the format/style of past examinations. However, students shall be given 6 hours to complete each examination. This time limit is intentionally generous. The expectation is that the examination can be successfully finished in less time (sometimes significantly less). Students shall be given all the questions of the entire examination at the beginning of the examination.

Passing criteria: In order to pass the examination the student needs to obtain at least 75% overall, *i.e.,* the minimum passing mark shall be $\frac{3}{4}(W_1 + W_2 + W_3)$.

5. The style of the examination (*i.e.,* the number of parts and their description explicitly mentioning the values for N_i, M_i and W_i), and the explicit passing criteria shall be provided on the examination paper (in a manner understandable to a student). The Examining Committee may provide additional instructions, recommendations, expectations and/or any other information on the examination paper as well.

   All the information described above in this section shall be communicated to student(s) 1 (one) working day before the examination date. (For example, the first page of the examination can contain this information and be set up by the Examining Committee so that it could be photocopied and given to student(s).)

6. The Examining Committee assigns the task of detailed grading of the various problems on each examination paper among the Committee members as it sees fit.

   In particular, individual examiners may be assigned particular questions to grade; or every examiner may be asked to grade the entire examination in detail. However, regardless of the particular methods used, in the end it is the responsibility of every Committee member to read the entire paper, and to form at least a general opinion of the performance of the student.

7. No member of the Examining Committee shall write on a student's original examination paper. Copies of the paper shall be made for each examiner's personal use.

8. After all the papers have been graded the Examining Committee shall have a meeting (virtual meeting or "meeting by email" is allowed) to decide on the marks to be given for each question on each of the examination papers. The Examining Committee decides how these marks are to be given as it sees fit (for example, by rounding the averages of the marks given by each examiner, by re-marking each problem in a committee, etc.). Once this marking is complete,

the Examining Committee shall use the "pass criteria" already set and indicated on the examination paper to determine if the student passes or fails the examination. The Coordinator shall write a report containing the mark decided upon by the Examining Committee, but not marks by individual examiners, of each question and comments (if any). The report must clearly indicate the result of the examination ("pass" or "fail"). It cannot contain statements of the sort "passed/failed notwithstanding". The report shall be given to the Associate Head (Graduate Studies) who will forward it the candidate as well as the Department for storage.

9. During the appeal period, neither the Coordinator nor the Committee shall communicate with the students, their advisors or anybody else other than the Associate Head (Graduate Studies) and the Department Head on any matter pertaining to the examination unless the Associate Head or the Department Head instructs otherwise (in writing).
10. The Associate Head (Graduate Studies) or Department Head shall announce the results formally to each of the students by means of a letter on Department letterhead. A copy of each letter must be provided to the Department office for the student's files. In the case of a failure, the letter shall inform the student of the right to appeal the outcome (with a reminder about the internal two week deadline). In the case of a first failure, the letter shall inform the student of the possibility to rewrite the examination; and in the case of a second failure on a given examination that this result means that the student will be required by the Faculty of Graduate Studies to withdraw from the Ph.D. program.
11. In the case that a first failure has been reported, and after the appeal period is over, it is the responsibility of the Coordinator to ensure that the student receives general feedback on the reasons for the failure and some guidance in study and preparation for the second attempt.
12. After the result of the examination has been announced, the student may view a copy of his/her examination paper in the Department office, under supervision. Students shall not be allowed to make any copies of any of the examination papers at any time.
13. **Appeals.** If a student wishes to appeal the result of his/her examination, a formal appeal in writing must be addressed to and received by the Department Head within 10 (ten) working days of the announcement of the result to the student. Appeals based solely on disagreement of the allocation of marks for one or more questions will be automatically rejected.

B. Advice to students preparing for Ph.D. Comprehensive Examinations

1. A Graduate Comprehensive Examination in the Department of Mathematics is an examination of material that is normally taught in an undergraduate honours program. For greater clarity, any material that is covered by the syllabus may appear on a comprehensive examination irrespectively of whether or not it has been (or is) taught in undergraduate or graduate courses at the University of Manitoba or any other institution.

   Additionally, since it is impractical (if not impossible) to list all possible definitions/concepts in the syllabus, it should be understood that topics related to those in the syllabus are also covered. For example, if a section of a book is covered by a syllabus, and a topic is discussed in this section, then this topic may be covered by the examination even if that specific concept is not explicitly mentioned in the syllabus. It goes without saying that you should also know all the basic material that is a prerequisite for the topics that are covered by the syllabus.

   Even though comprehensive examinations are based on undergraduate material, they are being written by Ph.D. students, and so a more sophisticated level of understanding and presentation may be expected than might normally be demanded in relevant undergraduate courses.

2. Copies of old comprehensive examinations may be available from the Department office or from the Department website. However, you should have ABSOLUTELY NO EXPECTATIONS that future examinations will be identical or even remotely similar to any particular previous examination. Do not base your expectations for the examination on any particular previous examination.
3. Do not limit your study and preparation to one reference; reviewing all references provided with the syllabus will provide a much better preparation for different styles of questions.
4. Texts and old examinations from relevant undergraduate courses may be useful as a study supplement, but students should master the material in the syllabus of the comprehensive examinations which follow.
5. Do not expect any Faculty member to “teach” you the subject matter of the examination (outside the regular coursework). You are admitted to our Ph.D. program on the basis that you are (mostly) prepared for work at this level, including being prepared (or having the mathematical maturity to be able to prepare) for the Comprehensive Examinations. You should be able to prepare for these examinations by reviewing of your earlier studies, and by self-studying of any subjects that you have missed.
6. Faculty members may be able to advise you on relevant study plans, be helpful with specific difficulties that you may have, and so on. The Associate Head (Graduate Studies) can help clarify the regulations; the Coordinator of the Examining Committee may help you interpret the details of the Examination Syllabus; and members of Faculty working in the general area of the Examination are often available for help with specific questions.
7. Expect a mixture of theoretical and practical problems, sometimes in the same question. Classifying questions as theoretical or practical is in many cases arbitrary, and a comprehensive examination in any case may (and often will) cover the entire range of styles of questions suitable to the subject area.
8. Corrective action can be taken at any time if mistakes in grading and/or questions are found, even after expiration of the appeal period.

Algebra Comprehensive Examination Syllabus

1. **Linear Algebra.**

- ▷ Vector spaces over $\mathbb{R}$, $\mathbb{C}$, finite fields and general fields.
- ▷ Subspaces. Linear independence, generating (spanning) sets, bases. Dimension. Infinite dimensional vector spaces. Direct sums. Every vector space has a basis (application of Zorn's Lemma or equivalent).
- ▷ Matrices. Matrix algebra. Inverses of matrices. Singular matrices. Determinants. Rank of matrices. Equivalent conditions for nonsingularity. Row space, column space, null space, rank of matrices and their interrelations.
- ▷ The theory of systems of linear equations. Elementary matrices. Gaussian Elimination. The reduced row echelon form. Solution space of a homogeneous system, particular and general solutions of an arbitrary system.
- ▷ Linear transformations. Representations by matrices in the finite dimensional case. Fundamental subspaces associated with a transformation (kernel, range). Fundamental subspaces associated with a matrix and their connection to linear transformations. Dimension theorem.
- ▷ Linear operators on finite and infinite dimensional spaces.
- ▷ Linear functionals. Duals of both finite and infinite dimensional spaces.
- ▷ Eigenvalues and eigenvectors of a linear operator; of a matrix. Characteristic equation of a matrix. The minimal polynomial. Diagonalization of matrices (operators) and applications. Invariant subspaces. Cayley-Hamilton theorem.
- ▷ Inner product spaces. Cauchy-Schwarz theorem. Orthogonal complements. The Gram-Schmidt algorithm. Adjoint of a linear operator. Unitary, self-adjoint, normal and orthogonal operators and their matrices. Diagonalization and self-adjoint operators.
- ▷ Jordan canonical form.
- ▷ Bilinear forms and their representations. Quadratic forms. Positive definite and positive semidefinite bilinear forms.

2. **Group Theory.**

- ▷ Semigroups, monoids, groups. Left and right inverses, units, idempotents. Order of an element of a group, order of a group. Cyclic groups. Permutation groups, Cayley's representation theorem. Dihedral groups, matrix groups, examples of symmetry groups. Group actions, stabilizer, orbit.
- ▷ Subgroups, cosets, normal subgroups. Lagrange's theorem.
- ▷ Homomorphisms and endomorphisms. Normal groups and factor groups. Isomorphism theorems. Simple groups. Commutator subgroup. Automorphisms, inner automorphisms.
- ▷ Centre, centralizer, normalizer, conjugate subgroups. Class equation.
- ▷ Sylow Theorems. p-groups, Sylow p-subgroups. Applications of the Sylow theorems. Nilpotent groups.
- ▷ Direct product (internal and external) of groups.
- ▷ Free abelian groups. The structure of finitely generated abelian groups, of finite abelian groups.

- ▷ Presentations (generators and relations). Free Groups.
- ▷ Normal series, composition series. Solvable groups.

3. **Ring Theory.**

- ▷ Rings, domains, fields, division rings. The ring of endomorphisms of an abelian group. Matrix rings. Characteristic of a ring. Subrings. Ideals. Simple rings. Homomorphisms and quotient (factor) rings. Isomorphism theorems.
- ▷ Embedding an integral domain in a field (quotient field construction).
- ▷ Ideals of a commutative ring. Prime ideals, Maximal ideals. Chinese Remainder Theorem for commutative rings.
- ▷ Unique Factorization Domains, Principal Ideal Domains, Euclidean Domains and the implications between them. Factorization theory in integral domains.
- ▷ Polynomial rings. Extension of a homomorphism on R to a homomorphism on $R[x]$. Polynomial rings over integral domains, unique factorization domains, and fields. Hilbert Basis Theorem. Factorization theory in polynomial rings. Irreducibility and Eisenstein's Criterion.
- ▷ Chain conditions (ascending chain condition, descending chain condition), noetherian and artinian rings. Existence of maximal ideals and prime ideals (application of Zorn's Lemma or equivalent).

4. **Field Theory and Galois Theory.**

- ▷ Finite fields, order of finite fields. Prime subfields.
- ▷ Field extensions. Finite, algebraic, and finitely generated extensions. Primitive elements. Minimal polynomial. Roots of polynomials. Adding a root of a polynomial. Transcendental extensions and transcendance degree.
- ▷ Quadratic extensions, straightedge and compass constructions, constructible numbers.
- ▷ Splitting fields. Roots of unity. Cyclotomic extensions. Algebraically closed fields and their construction.
- ▷ Automorphism groups of field extensions. Separable and normal extensions. Subfield fixed by a group of automorphisms. Galois extensions and the Galois group of an extension. Fundamental Theorem of Galois Theory.
- ▷ Determining the Galois group of a polynomial. Galois group of the splitting field of a polynomial and permutations of the roots of the polynomial.
- ▷ Extensions by radicals. Equations "solvable by radicals". The general equation of degree n and its Galois group. Elementary symmetric polynomials and fundamental theorem of symmetric polynomials.

5. **Module Theory.**

- ▷ Modules over a ring. Abelian groups, vector spaces, and ideals as modules. Submodules, homomorphisms, quotient modules. Lattice of submodules. Isomorphism theorems. The ring of endomorphisms of a module.
- ▷ Product, direct sum, tensor product and their characterization by universal properties. The Hom functor. Short exact sequences and preservation of exactness.

▷ Injective, free, projective, and flat modules.

Suggested reading list:

- Primary references:

[1] P. M. Cohn, *Classic Algebra*. [QA 154.2 C63 2000].
[Chapters 1–9, 10.1–10.5, 10.8, 10.11, 11.1–11.5, Appendix 1]

[2] P. M. Cohn, *Basic Algebra: Groups, Rings, and Fields*. [QA 154.3 C64 2003].
[Sections 1.1, 1.2, 2.1–2.5, 4.1, 4.2, 4.4–4.8, 7]
The rest of this book is well beyond the scope of this syllabus.

- For an in-depth review of Linear Algebra and Galois Theory:

[3] K. Hoffman and R. Kunze, *Linear Algebra*. [QA 252 H67 1971].

[4] I. Stewart, *Galois Theory*. [QA 214 S74 2004].

- Additional reading:

[5] M. Artin, *Algebra*, 2nd ed. [QA 154.2 A77 2011].

[6] J. B. Fraleigh, *A First Course in Abstract Algebra*, 3rd ed. [QA 162 F7 1982].

[7] I. N. Herstein, *Topics in Algebra*. [QA 155 H4 1954].
(**Not** *Abstract Algebra*, which is more elementary)

[8] N. Jacobson, *Basic Algebra, Vol. I*. [QA 154.2 J32 1985 V.1].

[9] S. Lang, *Algebra*. [QA 154 L35 1965].
(**Not** *Undergraduate Algebra*, which is more elementary)

The exam questions are not selected from any one reference or references.

Analysis Comprehensive Examination Syllabus

The analysis comprehensive examination consists of two units: UNIT ONE, which is for the core material and UNIT TWO, which is for the specialized material. For UNIT TWO, each student shall choose 2 of the 3 subsections II.(a), II.(b) and II.(c) below that will be covered by the examination. The student must declare his/her choices when registering for the examination. UNIT TWO of the examination will only cover the two subsections that the student has previously chosen.

I. The core material

Students are expected to know all basic mathematical analysis/calculus (see, e.g., [11, Chapters 1-9] and corresponding sections from [10]). In addition, the following topics are required:

- Functions of bounded variation: Monotonic functions, Total variation, Functions of bounded variation expressed as the difference of increasing functions.
- Riemann and Riemann-Stieltjes integral: Definition and elementary properties, Integration by parts, Change of variable, Step functions as integrators, Reduction of Riemann-Stieltjes integral to a finite sum, Fundamental theorems of integral calculus.
- Sequences of functions: Pointwise and uniform convergence, Uniform convergence and continuity, Cauchy condition for uniform convergence, Uniform convergence of infinite series of functions, Uniform convergence and Riemann-Stieltjes integration, Uniform convergence of improper integrals, Uniform convergence and differentiation, Sufficient conditions for uniform convergence of a series (Weierstrass M-test, Dirichlet's test), Weierstrass approximation theorem.
- Inverse function and implicit function theorems, Theorems of Green, Gauss and Stokes.
- Fourier series: Orthogonal systems of functions, Theorem on best approximations, Fourier series of a function relative to an orthonormal system, Properties of the Fourier coefficients, Riesz-Fischer theorem.
- Basics of Hilbert spaces.
- Lebesgue Measure

  Lebesgue outer measure, Measurable sets and Lebesgue measure, G_δ and F_σ sets, Existence of a non-measurable set, Measurable functions, Egorov's Theorem.
- Lebesgue Integral

  Definition and properties of Lebesgue integral, Fatou's Lemma, Monotone convergence theorem, Dominated convergence theorem, Convergence in measure.
- Differentiation and Integration

  Vitali's cover and Vitali's theorem, Integral of the derivative of an increasing function, Differentiation of the indefinite integral, Absolutely continuous functions.
- Basic properties of complex numbers and analytic functions, Elementary analytic functions, Entire Functions, Cauchy-Riemann Theorem, Inverse Function Theorem.
- Contour Integrals, Cauchy's Theorem, Cauchy's Integral Formula, Cauchy's Inequalities, Liouville's Theorem, Fundamental Theorem of Algebra, Morera's Theorem, Maximum Modulus Theorem, Schwarz Lemma.
- Convergent series of analytic functions, Taylor series, Laurent series, Zeros of analytic functions and classification of singularities, Casorati-Weierstrass Theorem.

- ▹ Residue Theorem and evaluation of integrals by using the Residue Theorem.
- ▹ Identity Theorem, Rouche's Theorem and Principal of the Argument Theorem, Hurwitz's theorem, Open Mapping Theorem for analytic functions.

The topics listed above may be found in [10, Sections 6.1-6.7, 7.1-7.9, 7.19, 7.20, 9.1-9.11, 10.13, 11.1-11.6, 11.15], [14, Chapters 1-5 and Section 10.8] and [13, Chapters 1-6].

II. The specialized material

(a) Basic Functional Analysis

- ▹ Normed Spaces

  Definition of normed and Banach spaces, Subspaces, Quotient spaces, Linear operators and functionals, Equivalent conditions for continuity of linear operators, Spaces of bounded linear operators, Hahn-Banach theorem, Embedding of a normed space into its second dual, Reflexive Banach spaces, Closed graph theorem, Open mapping theorem, Baire category in metric spaces, Principle of uniform boundedness.
- ▹ L^p - Spaces

  The Minkowski and Holder inequalities, Completeness, Dual spaces of L^p - spaces (Riesz representation theorem).

The topics listed above in this subsection may be found in [14, Chapters 6 and 10].

(b) Abstract Measure and Integration

Measure spaces, Measurable functions, Simple functions, Definition of abstract integral, linearity with respect to integrand and additivity with respect to domain of integration, Convergence theorems (Monotone, Dominated, Fatou's lemma), Signed measures, Complex measures, Theorem of Hahn for signed measures, Absolute continuity and singularity for measures, total variation of a measure, Radon Nikodym theorem, Lebesgue decomposition theorem, Outer measures, Measures derived from outer measures, Extension of a measure on an algebra (semi-algebra) to a measure on a σ-algebra, Caratheodory theorem, Product measures and theorems of Fubini and Tonneli.

The topics listed above in this subsection may be found in [14, Chapters 11 and 12].

(c) Advanced Complex Analysis

Conformal mappings and linear fractional transformations, Spaces of analytic functions, Equicontinuity, Uniform convergence on compact sets, Completeness of families of analytic and meromorphic functions, Normal families, Montel's theorem, Riemann mapping theorem, Harmonic functions, Mean Value Property for harmonic functions, Maximum Modulus Theorems for harmonic functions, Solution of the Dirichlet problem, Harnack's inequality and Harnack's Theorem.

The topics listed above in this subsection may be found in [12, Chapters III, VII and X].

References:

[10] T. M. Apostol, *Mathematical analysis*, 2nd ed., Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974. [QA 300 A573 1974].

[11] R. C. Buck, *Advanced calculus*, 3rd ed., Waveland Pr. Inc., January 1, 2003. [QA 303 B917 2003].

[12] J. B. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978. [QA 331 C659 1978].

[13] J. E. Marsden and Michael J. Hoffman, *Basic complex analysis*, 3rd ed., W. H. Freeman and Company, New York, 1999. [QA 331 M378 1999].

[14] H. L. Royden, *Real analysis*, 3rd ed., Macmillan Publishing Company, New York, 1988. [QA 331.5 R6 1988].

[15] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987. [QA 300 R82 1987].

Combinatorics Comprehensive Examination Syllabus

The combinatorics comprehensive exam covers three mandatory areas: core combinatorial concepts, graph theory, and elementary design theory. For study purposes, the three areas may constitute an approximate 40:35:25 distribution of available marks.

Unless otherwise noted, the student is expected to know proofs of named theorems and of main theorems of each topic covered. The student is assumed to have a working knowledge of modular arithmetic, finite fields, and mathematical induction.

Core combinatorial concepts

Basic counting: Counting rules (product and sum rules), permutations, r-permutations, subsets, r-combinations, binomial coefficients, probability, sampling with replacement, occupancy problems (e.g., indistinguishable balls into distinguishable cells), Stirling numbers of the second kind, multinomial coefficients, binomial expansion, generating permutations and combinations, Gray codes, algorithms and complexity, pigeonhole principle, partitions of integers and Ferrers diagrams.
Relations: Binary relations, order (linear, partial), lexicographic order, linear extensions, chains, antichains, Sperner's lemma, Dilworth's theorem, interval orders, lattices, distributive and modular lattices, complementation in lattices, Boolean lattices.
Generating functions: Applications to distribution, composition, and partitions of integers, generalized binomial theorem, exponential generating functions, counting permutations, distinguishable balls into indistinguishable cells.
Recurrence relations: Fibonacci numbers, derangements, method of characteristic roots, recurrences involving convolutions, Catalan numbers (including counting lattice paths, triangulations), solution by generating functions.
Inclusion and exclusion: Principle of inclusion-exclusion, counting derangements, number of objects having exactly m properties.

References for core concepts

[RoTe:05] F. S. Roberts and B. Tesman, *Applied combinatorics*, 2nd ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2005. [QA164.R6.2005] See Chapters 2, Sections 4.1, 4.2, 4.3.2, Chapters 5–7.

[Brua:10] R. A. Brualdi, *Introductory combinatorics*, 5th ed., Pearson Prentice Hall, 2010. [QA164.B76.2010] See Sections 2.1–2.5, Sections 3.1, 3.2, 4.3, 4.5, 5.2, 5.3, 5.6, 6.1, 6.6, 7.1–7.4, 8.1, 8.3.

[vLWi:05] J. H. van Lint and R. M. Wilson, *A course in combinatorics*, 2nd ed., Cambridge University Press, Cambridge, UK, 2001. [QA164.L56.2001] See Chapters 6, 8.

[Came:94] P. J. Cameron, *Combinatorics: topics, techniques, and algorithms*, Cambridge University Press, Cambridge, UK, 1994 (with corrections, 1996). [QA164.C346.1994] See Sections 12.1–12.3, 20.1.

Graph theory

Basic graph theory: Graph, multigraph, labelled graph, degree, degree sequence, handshaking lemma, Havel–Hakimi theorem, isomorphism, subgraph, complete graph, regular graph, bipartite graph, complement, hypercubes, connected, cut vertex, block, bridge, distance, walk, adjacency matrix, counting walks, incidence matrix, trail, path, Dijkstra's algorithm, circuit, cycle, girth, radius, diameter, centre, eccentricity.
Trees: Their properties, chemical bonds and trees, Cayley's theorem for labelled trees, Prüfer sequences, minimum spanning trees, Kruskal's algorithm, Prim's algorithm.
Directed graphs: Digraph, tournament, strongly connected, transitive tournament, king, Robbin's theorem for one-way streets,
Connectivity: Vertex and edge connectivity, Menger's theorem (graph vertex version), Whitney's theorem.
Cycles and circuits: Eulerian circuits and trails in graphs and digraphs, Fleury's algorithm, DeBruijn sequences, rotating drum problem, Chinese postman problem, Hamiltonian paths and cycles, Travelling Salesman Problem, Ore's condition, Dirac's condition, independent set.
Planar graphs: Euler's formula for planar graphs, five regular polyhedra, planar duals, Four Colour Theorem (statement only), Kuratowski's and Wagner's theorems (statements only).
Colouring: Vertex colourings, chromatic number, Brooks' theorem, Nordhaus–Gaddum theorem, perfect graph, interval graph, chordal graph, edge colouring, chromatic index, Vizing's theorem (statement only), König's theorem (for chromatic index), Ramsey's theorem, small Ramsey numbers, upper and lower bounds for $R(k, k)$.
Matchings, covers, and flows: Flows in networks, max-cut min-flow algorithm, matchings, augmenting paths, perfect matchings, 1-factor, Hall's marriage theorem, systems of distinct representatives, stable marriages, Gale–Shapley algorithm, factorizations, Tutte's 1-factor theorem, Petersen's theorem for cubic bridgeless graphs, vertex covers, edge covers, Gallai's theorem (for independence and covering numbers), König–Egerváry theorem.

References for graph theory

[CLZ:11] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs and digraphs*, 5th ed., Chapman and Hall/CRC, 2011. See Chapters 1, 2, Section 3.1, 3.2, Chapter 4, Sections 6.1, 6.2, 8.1–8.4, 9.1, 10.1–10.3, 11.1, 11.2, 12.3.

[RoTe:05] See Sections 3.1–3.3.4, 3.5, 3.7, 3.8, 11.2.1, 11.3, 11.4.1, 11.5, 11.6.1, 12.1, 12.2, 12.4, 12.5, 12.8.1. 13.1, 13.2, 13.3, 13.4.

[Brua:10] See Sections 3.3, 11.1–11.5, 11.7, 12.1–12.3, 12.5, 12.6, Chapter 13.

Elementary design theory

BIBDs, dual and complement of a design, latin squares, problem of 36 officers, MOLS, latin squares over finite fields, finite projective planes, homogeneous coordinates, incidence matrices, t-designs, derived and residual designs, Hadamard matrices, resolvable designs, Kirkman's school girl problem, Steiner triple systems, Fisher's inequality, and statements of Bruck-Ryser-Chowla and Bruck-Ryser theorems.

References for design theory

[RoTe:05] See Sections 9.2.2, 9.2.3, 9.3, 9.4.1–9.4.7, 9.5.

[Brua:10] See Sections 1.4, Chapter 10.

[vLWi:05] See Chapter 19.

[Came:94] See Chapter 8.

Computational Mathematics Comprehensive Examination Syllabus

The Computational Mathematics comprehensive examination consists of 2 (two) Units X and Y.

- Unit X shall be based on the material from section I (Introductory Numerical Analysis).
- Unit Y shall be based on the material from 1 (one) of sections II (Numerical PDEs) or III (Approximation Theory).

The choice of Unit Y shall be declared by the student at the time of registration for the Computational Mathematics comprehensive examination.

I. Introductory Numerical Analysis

- ▷ Computer arithmetic and floating-point numbers, relative and absolute errors. Round-off-errors and their propagation.
- ▷ Polynomial Interpolation: Lagrange interpolation, Divided differences, Natural and clamped cubic spline interpolation.
- ▷ Numerical differentiation and integration: Richardson's extrapolation, trapezoid and Simpson's rules, Newton-Cotes formulas, Composite numerical integration, Gaussian quadratures.
- ▷ Least squares approximation, orthogonal polynomials approximation (Legendre and Chebyshev), and trigonometric polynomial approximation.
- ▷ Direct methods for solving linear systems: Gaussian elimination, Pivoting methods, Matrix factorization and special types of matrices.
- ▷ Iterative methods for solving linear systems: Jacobi and Gauss-Seidel iterative methods, Relaxation methods, Error bounds. Power method for eigenvalue problems.
- ▷ Numerical solutions for nonlinear equations and nonlinear systems: Bisection method, fixed point method, Newton's method, secant method. Error analysis.
- ▷ Numerical solution for initial value problems for ordinary differential equations: Single step method (Euler's method, Runge-Kutta methods), Multi-step methods (Adams' methods, predictor-corrector methods). Local truncation error. Consistency, stability and convergence of numerical methods.
- ▷ Numerical solution for boundary value problems of ordinary differential equations: Shooting methods for linear and nonlinear problems, finite difference methods for linear and nonlinear problems.

References:

The material covered in this section may be found in Chapter 2 of [1], and Sections 1.2, 2.1-2.4, 3.1, 3.3, 3.5, 4.1-4.4, 4.7, 5.2-5.4, 5.6, 5.9-5.10, 6.1, 6.2, 6.5, 6.6, 7.3-7.5, 8.1-8.3, 8.5, 9.3, 10.1, 10.2, 11.1-11.4. of [2].

[1] K. Atkinson and W. Han, *Elementary Numerical Analysis*, 3rd ed., Wiley-Interscience Publication, 2004. [QA 297 A83 2004].

[2] R. Burden and J. Faires, *Numerical Analysis*, 9th ed., Brooks/Cole, 2011. [QA 297 B84 2011].

II. **Numerical Analysis of PDEs**

- Explicit and implicit finite difference schemes for parabolic partial differential equations (PDEs) including Euler and Crank Nicholson schemes, convergence, consistency and stability (von Neumann stability, matrix stability).
- Finite difference schemes (second order, fourth order, upwind schemes) for elliptic PDEs, convergence, consistency and stability.
- Finite difference schemes for hyperbolic PDEs, convergence, consistency and stability, CFL condition.
- Finite element methods for elliptic PDEs: 1D and 2D finite elements, local and global interpolation error estimates, L^2 and energy norm error estimates, inverse estimates.
- Numerical Linear Algebra: Condition number, classical iterative methods and Krylov subspace methods, preconditioning.

References:

The material covered in this section may be found in Chapters 1 to 7 of [3], Sections 3.1-3.6, 4.1-4.4, 9.1-9.8, 10.1-10.5, 10.7 of [4] and Chapters 1, 3.1-3.5, 4 and 6 of [5].

[3] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, 1987. [TA 347 .F5 J66n 1987].

[4] R.J. LeVeque, *Finite Difference Methods for Ordinary and Partial Differential Equations: steady-state and time-dependent problems*, SIAM, Philadelphia, 2007. [QA 431 L548 2007].

[5] S.H. Lui, *Numerical Analysis of Partial Differential Equations*, Wiley-Interscience Publication, New York, 2011. [QA 377 L84 2011].

III. **Approximation Theory**

- Density, existence, uniqueness

  Kolmogorov theorem. Chebyshev polynomials. Haar (Chebyshev) systems. Theorems of Weierstrass, Korovkin, Stone, Müntz, Mergelyan.
- Polynomial approximation

  Polynomial partition of unity. Moduli of continuity. Direct theorem for approximation by polynomials. Moduli of smoothness. Whitney inequality. Bernstein polynomials, shape preserving properties. Trigonometric polynomial kernels: Dirichlet, Fejer, Jackson, Stechkin. Stechkin's theorem (without proof). Bernstein inequality for trigonometric polynomials. Inverse theorem for approximation by trigonometric approximation. Bernstein inequality for algebraic polynomials. Markov inequality. Influence of endpoints in polynomial approximation. Inverse theorem for approximation by algebraic polynomials. Lipschitz spaces.
- Non-polynomial approximation

  Splines. Euler's ideal spline. Kolmogorov's inequality for derivatives. Spline bases. B-splines. Estimates on errors of approximation by splines. Widths. K-functionals and applications.

References:

The material covered in this section may be found in [7] and in Sections 1.1-1.4, 2.1, 2.6, 2.7, 2.9, 3.1-3.6, 4.1, 4.5-4.7, 5.1-5.3, 5.5-5.9, 6.1-6.4, 7.1-7.3, 7.6, 7.7, 8.1-8.6, 10.1-10.3 of [6].

[6] R.A. DeVore and G.G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 303, Springer-Verlag, Berlin, 1993. [QA 221 D44 1993].

[7] K. Kopotun, A. V. Prymak, and I. O. Shevchuk, *Introduction to Approximation Theory: Study Guide*, 2011.

Differential Equations Comprehensive Examination Syllabus

Style of the examination:

The examination shall consist of 2 (two) units carrying equal weights: Unit I (Ordinary Differential Equations) and Unit II (Partial Differential Equations). For $X \in \{I, II\}$, unit X shall consist of up to three parts A, B and C.

a) Part A shall consist of $N_{1,X}$ mandatory questions, worth a total of $W_{1,X}$ marks (individual marks for each question may or may not be the same).

b) Part B (if any) shall provide a choice of $N_{2,X}$ questions worth $\omega_{2,X}$ marks each with the instruction that the student is allowed to attempt $M_{2,X}$ out of $N_{2,X}$ questions ($M_{2,X}$ shall be strictly less than $N_{2,X}$) with a clear indication of which $M_{2,X}$ questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_{2,X} := \omega_{2,X} M_{2,X}$).

c) Part C (if any) shall provide a choice of $N_{3,X}$ questions worth $\omega_{3,X}$ marks each with the instruction that the student is allowed to attempt $M_{3,X}$ out of $N_{3,X}$ questions ($M_{3,X}$ shall be strictly less than $N_{3,X}$) with a clear indication of which $M_{3,X}$ questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_{3,X} := \omega_{3,X} M_{3,X}$).

It is up to the Examining Committee to decide how many questions will be in each part of each unit and how many marks will each part be worth (*i.e.*, there are no restrictions on the values of $N_{i,X}$, $M_{i,X}$ and $W_{i,X}$), and there is no obligation to follow the format/style of past examinations.

Passing criteria: In order to pass the examination the student needs to obtain at least 75% overall, *i.e.*, the student will pass the examination if

$$\frac{\mu_I}{W_{1,I} + W_{2,I} + W_{3,I}} + \frac{\mu_{II}}{W_{1,II} + W_{2,II} + W_{3,II}} \geq \frac{3}{2},$$

where μ_I and μ_{II} are the marks obtained by the student in Units I and II, respectively.

Students are expected to know topics from analysis used in proofs in the suggested references; for example, complete metric spaces, Cauchy sequences, uniform continuity and uniform convergence, Lipschitz functions, etc.

Unit I. Ordinary Differential Equations

- **General theory of ordinary differential equations**
  Existence and existence and uniqueness theorems. Gronwall's lemma. Banach fixed point/contraction mapping theorem. Continuation of solutions. Maximal solutions. Continuous dependence on initial conditions. Invariance of sets under flows. Euler and 4^{th}-order Runge-Kutta methods.

- **Techniques for scalar first-order equations**
  Separable and linear equations. Substitution methods: homogeneous and Bernoulli equations; equations of the form $y' = f(ax + by + c)$. Exact equations and integrating factors.

- **Scalar n^{th}-order linear equations**
  Homogeneous linear equations (principle of superposition; linear independence; Wronskian;

fundamental set of solutions; general solution; solution to an initial value problem). Nonhomogeneous linear equations (general theory; variation of constants –also called variation of parameters). Methods to solve n^{th}-order linear equations with constant coefficients (characteristic equation; differential operators and the annihilator method; undetermined coefficients; Laplace transform –unit step, Dirac Delta and periodic functions, convolution). Methods to solve n^{th}-order linear equations with non-constant coefficients (reduction of order; Cauchy-Euler equations; series solutions about ordinary points and regular singular points). Limiting behaviour of solutions.

▷ **Linear systems of first-order differential equations**
Transformation of a scalar n^{th}-order linear equation into a system of n scalar first-order differential equations. Homogeneous systems (superposition principle; the vector space of solutions –fundamental set of solutions; fundamental matrix). Autonomous homogeneous systems (diagonalizable case; generalized eigenspace; matrix exponential; trichotomy –stable, unstable and centre subspaces). Nonhomogeneous systems (variation of constants).

▷ **Nonlinear systems**
Transformation of a scalar n^{th}-order equation into a system of n scalar first-order differential equations. Autonomous systems and fixed points. Linearization: the stable manifold and Hartman-Grobman theorems. α- and ω-limit sets. Stability, asymptotic stability, Lyapunov stability.

▷ **Basic bifurcation theory**
Normal forms of the saddle-node, transcritical and pitchfork bifurcations. Bifurcation diagrams. Transformation of the normal form of the Hopf bifurcation to polar coordinates. Study of the Hopf bifurcation in the polar plane, with construction of the Poincaré map.

▷ **Particular case of scalar 2$^{\text{nd}}$-order equations (planar systems)**
Phase plane analysis. Direction fields. Nullclines. Fixed points. Stability of fixed points. Classification of types of orbits/trajectories in the phase plane for systems with constant coefficients. Oscillations.

Coefficients of the 4$^{\text{th}}$-order Runge-Kutta method and tables of Laplace transforms will be provided, as needed.

Suggested references

[1] N.G. Markley, *Principles of Differential Equations*, Pure and Applied Mathematics (New York), Wiley-Interscience, Hoboken, NJ, 2004. [QA 371 M27 2004].

[2] R.K. Nagle, E.B. Saff, and A.D. Snider, *Fundamentals of Differential Equations and Boundary Value Problems*, Sixth edition, Pearson, 2012. [QA 371 N24 2012].

[3] L. Perko, *Differential Equations and Dynamical Systems*, Third edition, Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 2001. [QA 372 P47 2001].

Markley [1] is the main reference for Unit I; all material in it is to be known, with the exception of Sections 3.4, 4.5, 5.4, 7.4 and Chapter 6. Nagle, Saff and Snider [2] is a complementary resource with a presentation that is generally not as rigorous as in [1]; in the case where a result is present in both [1] and [2], it is the version in [1] that is used. In Perko [3], use Chapter 1, Sections 2.1 to 2.13, 3.4, 4.2 and 4.4.

Unit II. Partial Differential Equations

- ▷ **Classification**
  Order. Dimension. Linear and nonlinear problems. Homogeneous and nonhomogeneous equations. Parabolic, hyperbolic and elliptic equations.

- ▷ **Linear and quasilinear first-order equations**
  Existence and Uniqueness. Method of characteristics. Shock waves.

- ▷ **Linear second order equations in two independent variables**
  Existence and uniqueness theorems. Characteristics. Reduction to canonical form and general solutions. D'Alembert's formula for the wave equation. The Cauchy problem. The Riemann problem. Maximum principles.

- ▷ **Separation of variables - second order linear equations in two or more variables**
  Cartesian, polar, cylindrical, and spherical coordinates. Nonhomogeneous problems (reduction to a homogeneous problem; steady-state solutions; eigenfunction expansions).

- ▷ **Sturm-Liouville theory and boundary value problems**
  Regular, periodic and singular homogeneous Sturm-Liouville problems. General theory. Reduction of a general second order linear equation to self-adjoint form. Eigenfunction expansions (generalized Fourier series).

- ▷ **Green's functions**
  Dirichlet and Neuman boundary value problems and initial boundary value problems. Construction of Green's functions for specific domains.

- ▷ **Transform methods**
  Fourier, Finite Fourier, Hankel and Laplace transforms on bounded, semi-infinite and infinite domains. Application of these transforms to problems in Cartesian, polar, cylindrical and spherical coordinates.

Tables of transforms will be provided, as needed.

Suggested references

[4] T. Myint-U and L. Debnath, *Linear partial differential equations for scientists and engineers*, Fourth edition, Birkhäuser Boston Inc., Boston, MA, 2007.

[5] S. Salsa, *Partial differential equations in action. From modelling to theory*, Universitext, Springer-Verlag Italia, Milan, 2008.

Relevant material for Unit II is in Chapters 1 to 12 and 15 in [4] and Chapters 1 to 5 in [5], except for sections 2.4-2.6, 2.9, 2.10. Note that both books are available online through the University of Manitoba Libraries.

Topology Comprehensive Examination Syllabus

1: Basic Point-Set Topology

1. *Construction of topological spaces:* Topologies, bases, local bases, and subbases. Subspaces, product spaces (including infinite products), and quotient spaces. Metric topologies.
2. *Special maps:* Continuous, open, closed, and quotient maps. Homeomorphisms. Preservation of properties under maps.
3. *Connectedness:* Connectedness, path connectedness, (connected) components, and path components.
4. *Compactness:* Compactness, countable compactness, and sequential compactness. Tychonoff Theorem. Local compactness, and one-point compactification.
5. *Separation axioms:* T_0, T_1, T_2 (Hausdorff), regularity, T_3, complete regularity, $T_{3\frac{1}{2}}$ (Tychonoff), normality, T_4, and the relationships among them. Urysohn's Lemma, and Tietze's Extension Theorem.
6. *Countability axioms:* Separability, first countability, second countability, the Lindelöf property, and the relationships among them.
7. *Metric spaces:* Topological properties, as itemized in 1–6, and their characterizations in metric spaces.

Suggested reading

[1] James R. Munkres, *Topology*, 2nd ed., 2000. [QA 611 M82 2000].
§1-§35, §37, §43, §45, §48.

[2] Stephen Willard, *General Topology*, 1970. [QA 611 W55 1970].
§1-§9, §13-§19, §22-§27.

2: Algebraic Topology

1. Classification of 1-manifolds. Triangulation of 2-manifolds. Classification of compact connected 2-manifolds.
2. Isotopy. Homotopy. Retracts and deformational retracts.
3. The fundamental group. Simply connected spaces. Fundamental group of a circle.
4. Group presentations. Free groups. Tietze transformations. Free products of groups. Free products with amalgamation.
5. Seifert-van Kampen theorem and consequences.
6. Knot Groups. Link Groups.
7. Covering spaces. Lifting maps. Universal covers. Covering transformations. Fundamental groups and covering spaces. Relationship/correspondence between covering spaces and groups.

8. Applications:
   - ▷ Brouwer Fixed Point Theorem
   - ▷ Borsuk-Ulam Theorem

Suggested reading

[1] W. S. Massey, *Algebraic Topology: An Introduction*, 1977. [QA 612 M37].
Chapters 1 to 5.

[2] John M. Lee, *Introduction to Topological Manifolds*, 2000. [Ebook accessible through the University of Manitoba Libraries' online catalogue].
Chapters 1 to 12.