

Computational Mathematics Comprehensive Exam (April 20, 2015)

- This exam consists of two units.
 1. Numerical Analysis which consists of four questions worth 10 marks each. Answer all questions.
 2. Approximation Theory which consists of four questions worth 10 marks each. Answer all questions.
- To obtain full credit, intermediate steps in obtaining your answers must be given.
- You have 6 hours. No extra time will be allowed.
- The passing criterion is 80% in the exam. That is, $(M_1/80 + M_2/80) \times 100 \geq 80$, where M_i is your mark in unit i .

UNIT I: Numerical Analysis

Answer all questions.

1. Let $m > 1$ be an integer and $x_* \in \mathbb{R}$. Define $f(x) = (x - x_*)^m g(x)$, where $g \in C^2(\mathbb{R})$ and $g(x_*) \neq 0$. Find the linear rate at which Newton's method applied to f converges locally. (Let $e_n = x_n - x_*$. Recall that the iteration $\{x_n\}$ converges at linear rate $r < 1$ if $\lim_{n \rightarrow \infty} \left| \frac{e_{n+1}}{e_n} \right| = r$.) How would you modify your iteration so that your new iteration will converge at linear rate zero. Show, in fact, that it converges quadratically locally.
2. Let $h > 0$ and Ω be the triangle in the xy -plane with vertices $(0, 0)$, $(h, 0)$, $(0, h)$. Determine the values of $\alpha, a, b \in \mathbb{R}$ so that

$$\int_{\Omega} f(x, y) \, dx dy = \alpha f(a, b)$$

is exact for all linear polynomials of the form $f(x, y) = c_0 + c_1 x + c_2 y$. Suppose $f \in C^2(\bar{\Omega})$. Estimate the quadrature error

$$\left| \int_{\Omega} f(x, y) \, dx dy - \alpha f(a, b) \right|.$$

3. Meson is a charged particle that moves inside a nucleus under the Yukawa potential $V(\rho) = -\frac{e^{-\rho}}{\rho}$. The eigenvalue corresponding to the energy level λ for the meson is given by

$$-\psi''(\rho) + V(\rho)\psi(\rho) = \lambda\psi(\rho), \quad \psi(0) = 0 = \psi(\infty).$$

Make a change of variable $x = 1 - e^{-\rho}$ and then write the eigenvalue problem in variable x . Set up the usual second-order finite difference scheme for this eigenvalue problem, subdividing the domain into 100 equal subintervals (i.e., $h = .01$). Write down the equations in matrix form. Be sure you define all terms and specify the size of your system.

4. Suppose the following scheme is used to integrate the ODE $u'(t) = f(t, u(t))$ to time $T > 0$ with initial condition $u(0) = u_0$:

$$u_{n+1} = u_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}), \quad n \geq 1,$$

where $f_n = f(t_n, u_n)$. Assume $t_n = nh$, $n \geq 0$. Assuming that f is smooth, determine the order of consistency. Show that the method is stable. What method would you use to obtain u_1 to ensure that the maximum order of convergence of this method is attained? What is this maximum order of convergence?

UNIT II: Approximation Theory

Answer all questions.

1. Let $B_n(f, x)$ be the Bernstein polynomial of degree $\leq n$ for a function f on $[0, 1]$. Compute $B_n(F, x)$ for $F(x) = (x - 1/2)^2$ and $n \in \mathbb{N}$.
2. Does there exist a monotone continuous function f on $[0, 1]$ which is not Lipschitz for any $\alpha > 0$?
3. Let $n \in \mathbb{N}_0$, $f \in \mathbb{C}[a, b]$, $a \leq t_0 < t_1 < \dots < t_{n+1} \leq b$, and denote by P_m the space of algebraic polynomials of degree $\leq m$. Suppose that $p \in P_{n+1}$ interpolates f at the points t_i , $0 \leq i \leq n+1$, and $q \in P_{n+1}$ is such that $q(t_i) = (-1)^i$, $0 \leq i \leq n+1$. Prove that the polynomial

$$r(x) := p(x) - \frac{p^{(n+1)}(a)}{q^{(n+1)}(a)}q(x)$$

is the polynomial of best uniform approximation to f on $(t_i)_{i=0}^{n+1}$ from P_n , and find the error of approximation (in terms of t_i 's and f only).

4. Let $1 \leq p < \infty$ and let $E_n(f)_p$ be the rate of approximation of $f \in L_p[-1, 1]$ by algebraic polynomials of degree $\leq n$ in the L_p -norm. For $g(x) := x|x|$ and $n \in \mathbb{N}$, prove that $E_n(g)_p \leq cn^{-2}$, for some absolute constant $c > 0$.