THE UNIVERSITY OF MANITOBA

DATE: February 23, 2010
PAPER NO
DEPARTMENT & COURSE NO: 136.130
EXAMINATION: Vector Geom. and Lin. Algebra

NAME (print in ink): ____________________________________________
Student Number: ______________________________________________
Signature: ____________________________________________________
(I understand that cheating is a serious offense)

Identify your section:

☐ L05  K. Kopotun   5   TTh 10:00-11:15am
☐ L06  G. I. Moghadam 8   MWF 1:30-2:20pm
☐ L07  G. I. Moghadam 12  MWF 3:30-4:20pm
☐ L08  C. Platt   15   TTh 4:00-5:15pm
☐ L09  J. Sichler   E2   T 7:00-10:00pm

INSTRUCTIONS:

• This is a one hour exam consisting of 7 questions on 7 pages. Make sure you have a complete copy.

• Attempt as many problems as you can (bonus problem is optional).

• Note that you can achieve a score greater than 60.

• No texts, notes, calculators or other aids are permitted.

• Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

• Show all your work and justify your answers. Unjustified answers will receive LITTLE or NO CREDIT.

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1. Consider the following linear system of equations.

\[
\begin{align*}
  x + y - z &= 1 \\
  -2y + 4z &= 6 \\
  3x + y + z &= 9 \\
  3x - y + 5z &= 15
\end{align*}
\]

(ii) Using part (a), find all solutions of this system (i.e., determine the solution set).

(i) Find the reduced row-echelon form (RREF) of the augmented matrix. Make sure to indicate what row operations you are using.
2. Consider the system

\[
\begin{align*}
    x + y + 2z &= a \\
    2x + by + 4z &= 1
\end{align*}
\]

In each case, determine all \(a\) and \(b\) which give the indicated number of solutions, if possible. If no such \(a\) and \(b\) exist, give the reason why not.

(i) no solutions

(ii) exactly one solution

(iii) exactly two solutions

(iv) infinitely many solutions
3. Which of the following statements are true. Start with “true” or “false” and then justify your answer.

(i) $\det((2A)^{-1}A^T(2A^T)) = \det(A)$ for all $2006 \times 2006$ invertible matrices $A$.

(ii) A product of elementary matrices may be singular (i.e., non-invertible).

(iii) Let $A = [a_{ij}]$ be a $2006 \times 2006$ matrix such that

$$a_{ij} = \begin{cases} 1, & \text{if } i \leq j, \\ 0, & \text{if } i > j. \end{cases}$$

In other words, all entries below the main diagonal of $A$ are equal to zero, and all other entries are equal to 1. Then $A$ is singular (i.e., non-invertible).
4. Let $A$ be a $2 \times 2$ matrix. Suppose that the matrix $B$ is obtained from $A$, and the matrix $I$ is obtained from $B$ by the following elementary row operations:

$$
\begin{align*}
A & \xrightarrow{\text{add 3 times row 2 to row 1}} B & B & \xrightarrow{\text{multiply row 2 by (-4)}} I
\end{align*}
$$

(i) Find elementary matrices $E_1$ and $E_2$ such that $E_1A = B$ and $E_2E_1A = I$

(ii) Express $A$ as a product of elementary matrices
5. Suppose that $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

(i) Find $A^{-1}$.

(ii) Let $C = \begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$. Find a matrix $B$ such that $AB = C$, and explain why your answer is the only one possible.
6. Let $A$ be a $4 \times 4$ invertible matrix and let

$$B = \text{adj}(A) = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 4 & 3 & 2 \\
0 & -2 & -1 & 2
\end{bmatrix}$$

(i) Calculate $\det(B)$.

(ii) Determine the value of $\det(A)$.
7. Suppose that $A$ is a $6 \times 6$ matrix such that $A^2 + I = 0$. Find all possible values of $\det(A)$.

[4] Bonus. **WARNING:** This is an optional BONUS question. Attempt it only if you have enough time and have solved all other problems.

Let $W$ be a $2006 \times 2006$ matrix such that $(W - I)^T = 2W$. Prove that $W$ is symmetric.