THE UNIVERSITY OF MANITOBA

DATE: April 13, 2009
DEPARTMENT & COURSE NO. MATH 1300
EXAMINATION: Vector Geometry & Linear Algebra
PAPER NO: 97

NAME: (Print in ink) _____________________________________________________________

STUDENT NUMBER: (in ink) _____________________________________________________

EXAMINATION ROOM __________________________________ SEAT NO. _______________

SIGNATURE: (in ink) ____________________________________________________________
(I understand that cheating is a serious offense)

Please indicate your instructor and section by placing a check mark in the appropriate box below.

☐ A01    D. Kelly  Tu, Th  10:00 am - 11:15 am
☐ A02    M. Doob  M, W, F  1:30 pm - 2:20 pm
☐ A03    J. Chipalkatti  M, W, F  3:30 pm - 4:20 pm
☐ A04    J. Arino  Tu, Th  4:00 pm - 5:15 pm

INSTRUCTIONS TO STUDENTS:

This is a 2 hour exam. Please show your work clearly.

No texts, notes, or other aids are permitted. Calculators, cell phones or electronic translators are also not permitted.

This exam has a title page, 6 pages of questions and also 2 blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but do not remove the staple.

The value of each question is indicated in the left-hand margin beside the statement of the question. The total value of all questions is 120.

Answer all questions on the exam paper in the space provide beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

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1. (14) Let $P(-1,1,1)$, $Q(1,2,3)$ and $R(2,1,0)$ be points in $\mathbb{R}^3$.

(a) Find the equation of the plane in $\mathbb{R}^3$ containing $P$, $Q$ and $R$.

(b) Find the equation of line $L$ in $\mathbb{R}^3$ passing through $P$ and $Q$.

(c) Find the area $A$ of the triangle determined by $P$, $Q$ and $R$.

(d) Find the point of intersection of $L$ and the $xy$-plane.
2. (15) Let

\[ A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

(a) Find the determinant of \( A \).

(b) Find \( A^{-1} \).

(c) Find all solutions \( x \) to

\[ Ax = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \]

(d) Find the adjoint of \( A \).
3. (24) Let

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \]

Evaluate the following matrices:

(a) \( A^T \)

(b) \( A^{-1} \)

c) \((A^{-1})^T\)

d) \((A^T)^{-1}\)

e) \((A^{-1})^2\)

(f) \((A^2)^{-1}\)

4. (10)

(a) Calculate \( \text{proj}_v u \), the projection of \( u \) along \( v \), where \( u = (2, -2, 3) \) and \( v = (2, 1, 3) \).

(b) Find a unit vector in the direction of \( w = (4, -3, 2, 1) \).

(c) Find all values \( t \) so that vectors \((2, t, -3, 6)\) and \((4, 4, 7, 1)\) are orthogonal.
5. (16) Let \( u = (1,0,1) \) and \( v = (0,1,1) \).

(a) Find a vector \( w \) in \( \mathbb{R}^3 \) that is not in the span of \( \{u, v\} \). You must justify your answer.

(b) Find a vector \( w \) in \( \mathbb{R}^3 \) so that \( \{u, v, w\} \) is a basis for \( \mathbb{R}^3 \). You must justify your answer.

(c) Let \( \{u, v, w\} \) be the basis you gave in part (b). Find real numbers \( a, b \) and \( c \) so that \( (25, 13, -17) = au + bv + cw \).

(d) Show that \( C = \begin{bmatrix} 11 & 17 \\ 1 & 13 \end{bmatrix} \) is in the span of \( \{A, B\} \) where \( A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix} \).

6. (4) In this question, each answer is a number.

(a) The smallest possible value for the dimension of the null space of a \( 17 \times 23 \) matrix is ________.

(b) The largest possible value of the dimension of the column space of a \( 17 \times 23 \) matrix is ________.

(c) Let \( A \) be a \( 17 \times 23 \) matrix whose null space has dimension 15. The dimension of the row space is ________.

(d) the span of \((1, 1, 1, 1, 1), (1, 1, 0, 0, 1) \) and \((0, 0, 1, 1, -1) \) has dimension ________.
7. (18) The matrix

\[ A = \begin{bmatrix} 2 & -10 & 1 & 1 & 22 & -2 & 0 \\ 4 & -20 & 1 & 1 & 36 & 0 & 18 \\ 1 & -5 & 2 & 2 & 23 & 3 & 26 \\ 4 & -20 & -2 & -2 & 12 & 3 & 19 \end{bmatrix} \]

has reduced row echelon form

\[ R = \begin{bmatrix} 1 & -5 & 0 & 0 & 7 & 0 & 3 \\ 0 & 0 & 1 & 1 & 8 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

(a) Find a basis for the null space of \( A \).

(b) Find a basis for the row space of \( A \).

(c) Find a basis for the column space of \( A \).

(d) The dimension of the null space of \( A \) is ______

(e) The dimension of the row space of \( A \) is ______

(f) The dimension of the column space of \( A \) is ______
8. (12) In each part of this question a subset $W$ of the vector space $\mathbb{R}^5$ is defined. State whether or not $W$ is a subspace of $\mathbb{R}^5$. Justify your answer.

(a) $W = \{(0, a, 2a + 3b, 0, 5a - 6b + 3c) \mid a, b, c \in \mathbb{R}\}.$

(b) $W = \{(0, a, 2a + 3, 0, a + b + c) \mid a, b, c \in \mathbb{R}\}.$

(c) $W = \{(a, b, c, d, e) \mid 7a + 5b + 3c + 2d + e = 0\}.$

9. (7) Indicate whether each statement is true or false:

(a) Whenever a finite set $S$ spans a vector space $V$, then there is a subset of $S$ which is a basis of $V$ ........................................... □ True □ False

(b) If a vector $v$ is not in the span of $\{u_1, u_2, u_3\}$, then the set $\{u_1, u_2, u_3\}$ is linearly independent ........................................... □ True □ False

(c) If $V$ and $W$ are subspace of some $\mathbb{R}^n$, then the set $\{v + w \mid v \in V \text{ and } w \in W\}$ is a subspace of $\mathbb{R}^n$ ........................................... □ True □ False

(d) There is a basis for $\mathbb{R}^6$ which contains the following three vectors: $(5, 3, 4, 7, 5, 6), (0, 6, 5, 4, 3, 7)$ and $(0, 0, 2, 10, 3)$ ........................................... □ True □ False

(e) $\det(-A) = -\det(A)$ ........................................... □ True □ False

(f) $\det(AB) = \det(A) \det(B)$ ........................................... □ True □ False

(g) $\det(A + B) = \det(A) + \det(B)$ ........................................... □ True □ False