THE UNIVERSITY OF MANITOBA

December 14, 2004, 1:30 p.m.
PAPER NO.: 250
DEPARTMENT & COURSE NO: Mathematics - 136.310
EXAMINATION: Mathematical Methods for Engineers 3

FINAL EXAMINATION
PAGE NO.: 1 of 13
TIME: 3 Hours
EXAMINERS: J. J. Williams and D. W. Trim

Name
(PRINT):

Name
(Signature):

Student
Number:

Seat Number:

Examination Room: Gold Gym - Frank Kennedy

Please indicate your section - slot - instructor:

☐ L01, slot 11, J. J. Williams

OR

☐ L02, slot 12, D. W. Trim

DO NOT WRITE IN THESE COLUMNS

<table>
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<tr>
<th>QUESTION NUMBER</th>
<th>MARK</th>
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INSTRUCTIONS:

(i) Fill in all of the above requested information.
(ii) Calculators, notes or other aids are NOT allowed.
(iii) PLEASE CHECK THAT YOUR BOOKLET IS COMPLETE: this examination consists of 8 questions for a total of 100 marks.

NOTE: references and a summary sheet on Fourier Series are printed at the end of this exam; see pages 12 - 13.

(iv) Attempt all questions; please show your work and simplify all answers.
(v) If space is insufficient, please use the back of the previous page, i.e., the page that faces that question, to show additional work.
(vi) You may remove the references and summary sheet on Fourier Series on pages 12 - 13. Otherwise, do NOT remove any other pages from this booklet, and hand in this entire question paper at the conclusion of the examination.

SEE NEXT PAGE ➔
1. Evaluate the line integral:

\[ \int_C \sqrt{x} \, dx - z \, dy + (x + y) \, dz, \]

where \( C \) is the curve: \( y - z = 1, \ x + 2 = z^2 \) from the point \( (2, 3, 2) \) to the point \( (7, 4, 3) \).
2. Evaluate the line integral:

\[
\int_C (2xy - z^2)dx + (x^2 + 1)dy - 2xzdz,
\]

where \( C \) lies in the first octant along the intersection of the surfaces:
\( z^2 = x^2 + y^2 \) and \( y + z = 2 \) from the point \((0, 1, 1)\) to the point \((2, 0, 2)\).
(10) 3. Evaluate the surface integral:

\[ \iint_S \mathbf{F} \cdot \mathbf{n} \, dS, \]

where \( S \) is that part of the paraboloid \( z = 4 - x^2 - y^2 \) that lies above the xy-pane, \( \mathbf{n} \) is the unit upper normal to \( S \), and \( \mathbf{F} = xz\mathbf{i} + yz\mathbf{j} \).
(12) 4. (a) Show that \( x = 0 \) is an ordinary point of the differential equation:

\[
(x^2 - 1) \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} - 5y = 0.
\]

(b) Suppose a solution of the differential equation in part (a) can be represented in terms of its Maclaurin series:

\[
y = \sum_{n=0}^{\infty} c_n x^n.
\]

Find a recurrence relation for the coefficients, \( c_n \), and simplify as much as possible.

Do NOT attempt to iterate this formula; i.e., do NOT solve for the \( c_n \).
5. When a solution of the differential equation:

\[(x^2 + 3)y'' + 2xy' - 2y = 0\]

is represented in terms of its Maclaurin series, \( y = \sum_{n=0}^{\infty} c_n x^n \), its coefficients, \( c_n \), must satisfy the recurrence relation:

\[c_{n+2} = \frac{-(n-1)}{3(n+1)} c_n, \quad n \geq 0.\]

You may assume the above recurrence relation. Do NOT show this.

(a) Use this recurrence relation to find the general solution of the differential equation and simplify as much as possible. Write any infinite series using the sigma notation.

(b) Determine a guaranteed open interval of convergence for the solution.
(12) 6. (a) Calculate the Fourier Sine series of \( f(x) \) on the interval \([0, 3]\), where 
\[ f(x) = (3 + x), \quad 0 \leq x \leq 3 \] and simplify your answer.

(b) Using the "graph paper" below, draw the graph of the Fourier sine series of \( f \), as found in part (a), for \(-9 \leq x \leq 9\), and indicate clearly (on the graph) the scale on the \( y \)-axis and the value at each point of discontinuity:

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<tbody>
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<td>-9</td>
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<td>-3</td>
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<td>6</td>
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<td>9</td>
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</tbody>
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SEE NEXT PAGE
7. A thin laterally-insulated rod has diffusivity $7$ and length $4$.

The equation for heat conduction in this rod is given by:

$$7 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 4, \quad t > 0,$$

where $u(x, t)$ is the temperature at the point $x$, at time $t$.

The boundary conditions are given by:

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(4, t) = 0, \quad \text{for} \quad t > 0.$$

Initially, (at time $t = 0$), the rod has a temperature of $(10 - x)$ at the point $x$, i.e., $u(x, 0) = 10 - x$, for $0 < x < 4$.

(a) Solve the problem for $u(x, t)$ by using separation of variables.

Show all the details of your calculations and simplify your answer.

(b) What is the physical interpretation of the boundary conditions?

(c) Find the limit $u(x, t)$ as $t \to \infty$ (if possible), and give a brief physical interpretation of your answer.

Note: there is extra space on the next page to continue your work.
8. This question asks you ONLY to fill in the table below.
A vibrating string has length 6 (m) and at equilibrium it lies along the \(x\)-axis from \(x = 0\) to \(x = 6\). The velocity of propagation of the waves along the string is 4 (m/s).
Let \(u(x, t)\) be the displacement (in metres) from the \(x\)-axis. The left end is attached to a frictionless ring sliding on the \(x\)-axis and the right end is attached to the \(x\)-axis. Initially (at time \(t = 0\)), the string is in the shape of a parabola which is zero at both end-points with a maximum displacement of 2 m (at the centre, \(x = 3\) m). In addition, the initial velocity of each point of the string is 1 m/s.

> Write down the PDE, the boundary conditions, and the initial condition(s) that the displacement, \(u(x, t)\), must satisfy; include appropriate bounds on the variables. Do NOT solve the above problem for \(u(x, t)\).

<table>
<thead>
<tr>
<th>EQUATION</th>
<th>BOUND(S) ON VARIABLE(S)</th>
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<tbody>
<tr>
<td>PDE:</td>
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<tr>
<td>Boundary Condition Left end point:</td>
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<tr>
<td>Boundary Condition Right end point:</td>
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<td>Initial Condition(s):</td>
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Total = 100
Remarks: The theorems below are provided solely for your information.

Green's Theorem: \( \oint_C P \, dx + Q \, dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA, \)
where \( C \) is a closed curve in the \( x\)-\( y \) plane, enclosing a region \( R \);
i.e., \( C \) is the boundary of \( R \).

Divergence Theorem: \( \iiint_V \nabla \cdot F \, dV = \iint_S F \cdot \hat{n} \, dS, \)
where the surface \( S \) encloses the region \( V \), and \( \hat{n} \) is the unit outer normal to \( S \).

Stokes's Theorem: \( \oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot \hat{n} \, dS, \)
where \( S \) is a surface with \( C \) as its boundary, and \( \hat{n} \) is the unit normal to \( S \) such that the surface, \( S \), is on the left while traversing \( C \).

See SUMMARY OF FOURIER SERIES on next page ➤
SUMMARY OF FOURIER SERIES

I. FULL SERIES: Let \( f(x) \) be defined and piece-wise smooth on \( \{ x : a \leq x \leq b \} \). Let \( p = (b - a) / 2 \); the length of this interval = \( b - a = 2p \).

The Fourier series of \( f(x) \) on the interval \([a, b]\) is given by:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right), \quad a < x < b.
\]

\[
a_n = \frac{1}{p} \int_{a}^{b} f(x) \cos \left( \frac{n\pi x}{p} \right) \, dx, \quad b_n = \frac{1}{p} \int_{a}^{b} f(x) \sin \left( \frac{n\pi x}{p} \right) \, dx.
\]

The series converges to

\[
\begin{cases} 
\frac{1}{2} [f(x^+) + f(x^-)], & \ a < x < b \\
\frac{1}{2} [f(a^+) + f(b^-)], & \ x = a, b
\end{cases}
\]

its period = \( 2p = b - a \).

II. HALF-RANGE EXPANSIONS: Let \( f(x) \) be piece-wise smooth on \( \{ x : 0 \leq x \leq L \} \).

(a) FOURIER COSINE SERIES: Extend \( f \) to an even function for \( -L \leq x \leq L \).

Period = 2L. Series equals \( f(0) \) at \( x = 0 \), and \( f(L) \) at \( x = L \).

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L; \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx.
\]

(b) FOURIER SINE SERIES: Extend \( f \) to an odd function for \( -L \leq x \leq L \).

Period = 2L. Series equals 0 at \( x = 0, L \).

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 < x < L; \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \left( \frac{n\pi x}{L} \right) \, dx.
\]

III. FORMULAE: (\( k \) is a non-zero constant):

\[
\begin{align*}
\int \cos kx \, dx &= \frac{1}{k} \sin kx, \\
\int \sin kx \, dx &= -\frac{1}{k} \cos kx, \\
\int x \cos kx \, dx &= \frac{1}{k^2} \cos kx + \frac{x}{k} \sin kx, \\
\int x \sin kx \, dx &= \frac{1}{k^2} \sin kx - \frac{x}{k} \cos kx, \\
\int x^2 \cos kx \, dx &= \frac{x^2}{k} \sin kx + \frac{2x}{k^2} \cos kx - \frac{2}{k^3} \sin kx, \\
\int x^2 \sin kx \, dx &= -\frac{x^2}{k} \cos kx + \frac{2x}{k^2} \sin kx + \frac{2}{k^3} \cos kx.
\end{align*}
\]

For any integer \( n \):

\[
\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n, \quad \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n, \quad \cos\left(\frac{(2n+1)\pi}{2}\right) = 0.
\]

END OF EXAMINATION.

Have a happy and safe holiday !!!