Name (PRINT): _____

Name (Signature): _____

Student Number: _____

Seat Number: _____

Examination Room: Gold Gym - Frank Kennedy

Please indicate your section - slot - instructor:

☐ **L01, slot 11, J. J. Williams**

OR

☐ **L02, slot 12, D. W. Trim**

DO NOT WRITE IN THESE COLUMNS		
QUESTION NUMBER	MARK	MAX.
1		8
2		6
3		10
4		12
5		12
6		12
7		30
8		10
TOTAL		100

INSTRUCTIONS:

(i) Fill in all of the above requested information.

(ii) Calculators, notes or other aids are **NOT** allowed.

(iii) **PLEASE CHECK THAT YOUR BOOKLET IS COMPLETE:** this examination consists of 8 questions for a total of 100 marks.

NOTE: references and a summary sheet on *Fourier Series* are printed at the end of this exam; see pages 12 - 13.

(iv) Attempt **all** questions; please show your work and **simplify** all answers.

(v) *If space is insufficient, please use the back of the* ***previous*** *page, i.e., the page that faces that question, to show additional work.*

(vi) You may remove the references and summary sheet on Fourier Series on pages 12 - 13. **Otherwise, do NOT remove any other pages from this booklet, and hand in this entire question paper at the conclusion of the examination.**

(8) 1. Evaluate the line integral:

$$\int_C \sqrt{x}\,dx - z\,dy + (x+y)\,dz,$$

where C is the curve: $y - z = 1$, $x + 2 = z^2$ from the point $(2, 3, 2)$ to the point $(7, 4, 3)$.

(6) 2. Evaluate the line integral:

$$\int_C (2xy - z^2)dx + (x^2 + 1)dy - 2xzdz,$$

where C lies in the first octant along the intersection of the surfaces: $z^2 = x^2 + y^2$ and $y + z = 2$ from the point $(0, 1, 1)$ to the point $(2, 0, 2)$.

VALUES

(10) 3. Evaluate the surface integral:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS,$$

where S is that part of the paraboloid: $z = 4 - x^2 - y^2$ that lies above the xy-pane, $\hat{\mathbf{n}}$ is the unit upper normal to S, and $\mathbf{F} = xz\hat{\mathbf{i}} + yz\hat{\mathbf{j}}$.

(12) 4. (a) Show that $x = 0$ is an ordinary point of the differential equation:

$$\left(x^2 - 1\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 5y = 0.$$

(b) Suppose a solution of the differential equation in part (a) can be represented in terms of its Maclaurin series:

$$y = \sum_{n=0}^{\infty} c_n x^n.$$

Find a **recurrence relation** for the coefficients, c_n, and *simplify* as much as possible.

Do **NOT** attempt to iterate this formula; i.e., do **NOT** solve for the c_n.

(12) 5. When a solution of the differential equation:

$$\left(x^2+3\right)y''+2xy'-2y=0$$

is represented in terms of its Maclaurin series, $y=\sum_{n=0}^{\infty} c_n x^n$, its coefficients, c_n, must satisfy the recurrence relation:

$$c_{n+2}=\frac{-(n-1)}{3(n+1)}c_n, \quad n\geq 0.$$

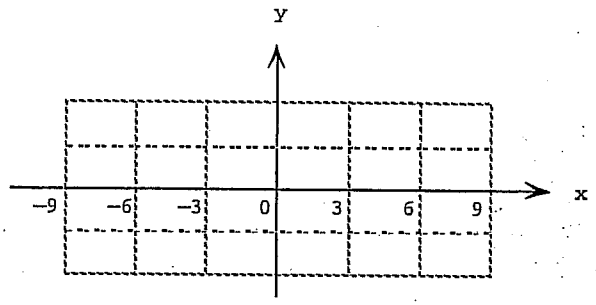
You may assume the above recurrence relation. Do *NOT* show this.

(a) Use this recurrence relation to find the general solution of the differential equation and *simplify* as much as possible. Write any infinite series using the *sigma* notation.

(b) Determine a **guaranteed** open interval of convergence for the solution.

(12) 6. (a) Calculate the Fourier **Sine** series of $f(x)$ on the interval $[0, 3]$, where $f(x) = (3 + x)$, $0 \le x \le 3$ and ***simplify*** your answer.

(b) Using the "graph paper" below, draw the graph of the Fourier sine series of f, as found in part (a), for $-9 \le x \le 9$, and indicate clearly (on the graph) the scale on the y–axis and the value at each point of discontinuity:



(30) 7. A thin laterally-insulated rod has diffusivity 7 and length 4.
The equation for heat conduction in this rod is given by:

$$7\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < 4, \quad t > 0,$$

where $u(x, t)$ is the temperature at the point x, at time t.
The boundary conditions are given by:

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(4,t) = 0, \text{ for } t > 0.$$

Initially, (at time $t = 0$), the rod has a temperature of $(10 - x)$ at the point x, i.e., $u(x,0) = 10 - x$, for $0 < x < 4$.

(a) Solve the problem for $u(x, t)$ by using separation of variables.
Show all the details of your calculations and simplify your answer.

(b) What is the physical interpretation of the boundary conditions?

(c) Find the limit $u(x, t)$ as $t \to \infty$ (if possible), and give a brief physical interpretation of your answer.

Note: there is extra space on the next page to continue your work. ➔

VALUES

(10) 8. **This question asks you ONLY to fill in the table below.**

A vibrating string has length 6 (m) and at equilibrium it lies along the x–axis from $x = 0$ to $x = 6$. The velocity of propagation of the waves along the string is 4 (m/s).

Let $u(x, t)$ be the displacement (in metres) from the x–axis. The left end is attached to a frictionless ring sliding on the u–axis and the right end is attached to the x–axis. Initially (at time $t = 0$), the string is in the shape of a parabola which is zero at both end-points with a maximum displacement of 2 m (at the centre, $x = 3$ m). In addition, the initial velocity of each point of the string is 1 m/s.

➢ **Write down the PDE, the boundary conditions, and the initial condition(s) that the displacement, $u(x, t)$, must satisfy; include appropriate bounds on the variables. Do NOT solve the above problem for $u(x, t)$.**

	EQUATION	BOUND(S) ON VARIABLE(S)
PDE:		
Boundary Condition Left end point:		
Boundary Condition Right end point:		
Initial Condition(s):		

* *

Total = 100

References

Remark: The theorems below are provided solely for your information.

Green's Theorem: $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA,$

where C is a closed curve in the x-y plane, enclosing a region R; i.e., C is the boundary of R.

Divergence Theorem: $\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV,$

where the surface S encloses the region V, and $\hat{\mathbf{n}}$ is the unit outer normal to S.

Stokes's Theorem: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS,$

where S is a surface with C as its boundary, and $\hat{\mathbf{n}}$ is the unit normal to S such that the surface, S, is on the left while traversing C.

See SUMMARY OF FOURIER SERIES on next page ➤➤

SUMMARY OF FOURIER SERIES

I. FULL SERIES: Let $f(x)$ be defined and piece-wise smooth on $\{x : a \le x \le b\}$.
Let $p = (b-a)/2$; the length of this interval $= b - a = 2p$.

The Fourier series of $f(x)$ on the interval $[a, b]$ is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right), \quad a < x < b.$$

$$a_n = \frac{1}{p}\int_a^b f(x)\cos\left(\frac{n\pi x}{p}\right)dx, \qquad b_n = \frac{1}{p}\int_a^b f(x)\sin\left(\frac{n\pi x}{p}\right)dx.$$

The series converges to $\begin{cases} \frac{1}{2}\left[f(x^+) + f(x^-)\right], & a < x < b \\ \frac{1}{2}\left[f(a^+) + f(b^-)\right], & x = a, b \end{cases}$; its period $= 2p = b - a$.

II. HALF-RANGE EXPANSIONS: Let $f(x)$ be piece-wise smooth on $\{x : 0 \le x \le L\}$.

(a) **FOURIER COSINE SERIES:** Extend f to an *even* function for $-L \le x \le L$.
Period $= 2L$. Series equals $f(0)$ at $x = 0$, and $f(L)$ at $x = L$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad 0 \le x \le L; \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

(b) **FOURIER SINE SERIES:** Extend f to an *odd* function for $-L \le x \le L$.
Period $= 2L$. Series equals 0 at $x = 0, L$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0 < x < L; \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

III. FORMULAE: (k is a non-zero constant):

$$\int \cos kx\,dx = \frac{1}{k}\sin kx, \qquad \int \sin kx\,dx = -\frac{1}{k}\cos kx,$$

$$\int x\cos kx\,dx = \frac{1}{k^2}\cos kx + \frac{x}{k}\sin kx,$$

$$\int x\sin kx\,dx = \frac{1}{k^2}\sin kx - \frac{x}{k}\cos kx,$$

$$\int x^2\cos kx\,dx = \frac{x^2}{k}\sin kx + \frac{2x}{k^2}\cos kx - \frac{2}{k^3}\sin kx,$$

$$\int x^2\sin kx\,dx = -\frac{x^2}{k}\cos kx + \frac{2x}{k^2}\sin kx + \frac{2}{k^3}\cos kx.$$

For any integer n:

$$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n, \quad \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n, \quad \cos\left(\frac{(2n+1)\pi}{2}\right) = 0.$$

END OF EXAMINATION

Have a happy and safe holiday !!!