Name (PRINT):_____

Name (Signature):_____

Student Number:_____

Examination Room: E3-270 EIT Complex
(Seat #: 1 – 111)

Please indicate your section – lecture time - instructor:

☐ **A01, MWF 8:30 a.m., J. J. Williams**

OR

☐ **A02, MWF 9:30 a.m., D. Kalajdzievska**

DO NOT WRITE IN THESE COLUMNS

QUESTION NUMBER	MARK	MAX.
1		8
2		10
3		9
4		9
5		13
6		12
7		29
8		10
TOTAL		100

INSTRUCTIONS:

(i) Fill in all of the above requested information.

(ii) Calculators, notes or other aids are **NOT** allowed.

(iii) **PLEASE CHECK THAT YOUR BOOKLET IS COMPLETE:** this examination consists of 8 questions, printed on 14 pages, for a total of 100 marks.

NOTE: References, including a summary of Fourier Series, are printed at the end of this exam; see pages 13 - 14. There are two blank pages at the end of the examination for extra space; see pages 11 - 12.

(iv) Attempt **all** questions; please **show all of your work** and **simplify** all answers.

(v) You may detach the references and summary sheet on Fourier Series on pages 13 - 14.

Otherwise, do NOT detach any other page from this booklet, and hand in this entire question paper, including the references and summary sheet at the conclusion of the examination.

SEE NEXT PAGE ⇨

VALUES

(8) 1. Evaluate the line integral:

$$\int_C 27x\,ds,$$

where C is the curve $x = z^3$, $y = 2$ from the point $(1, 2, 1)$ to the point $(0, 2, 0)$.

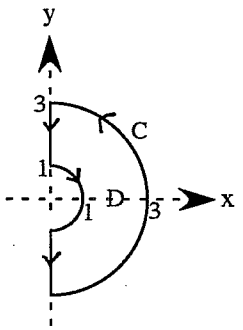
Simplify your final answer.

VALUES

(10) 2. Let $\mathbf{F} = (3xy)\hat{\mathbf{i}} + (2x^2)\hat{\mathbf{j}}$, and let C be the positively-oriented boundary of the semi-annular region D in the right half-plane between $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$. Note that C is a closed curve and consists of two semi-circles and two pieces of the y-axis.
See the diagram given.

(a) Is $\mathbf{F}$ a conservative vector field? Explain.

(b) Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.



VALUES

(9) 3. Evaluate the surface integral:

$$\oiint_S \left[(2x+y^2z)\hat{\mathbf{i}}+(x+y+z^2)\hat{\mathbf{j}}+(3x+z)\hat{\mathbf{k}}\right]\cdot\hat{\mathbf{n}}\,dS,$$

where S is the *closed* surface which encloses the volume in the first octant bounded by the cylinder: $x^2+y^2=4$, the three coordinate planes: $x=0$, $y=0$, and $z=0$, and its top is given by the slanted plane: $z=2+x$. $\hat{\mathbf{n}}$ is the unit inward pointing normal to S.
Note that S consists of five separate "sides": the curved "side" given by the cylinder, two flat sides ($x=0$ and $y=0$), a top ($z=2+x$) and a bottom ($z=0$).

VALUES

(9) 4. Consider the differential equation:

$$(2-x^2)y''-6xy'-4y=0.$$

A solution of the above differential equation can be represented by its Maclaurin series:

$$y=\sum_{n=0}^{\infty} c_n x^n.$$

Find a **recurrence relation** for the c_n and *simplify* it as much as possible.

Do NOT attempt to iterate this recurrence relation, i.e., do NOT solve for the c_n or calculate any solution for $y(x)$.

VALUES

(13) 5. When a solution of the differential equation:

$$(x^2+5)y''-6y=0$$

is represented in terms of its Maclaurin series, $y=\sum_{n=0}^{\infty} c_n x^n$, its coefficients, c_n, satisfy the recurrence relation: $c_{n+2}=\dfrac{-(n-3)}{5(n+1)}c_n, \quad n=0,1,2,3,\ldots$.

*Do **NOT** show this. You may assume the above recurrence relation.*

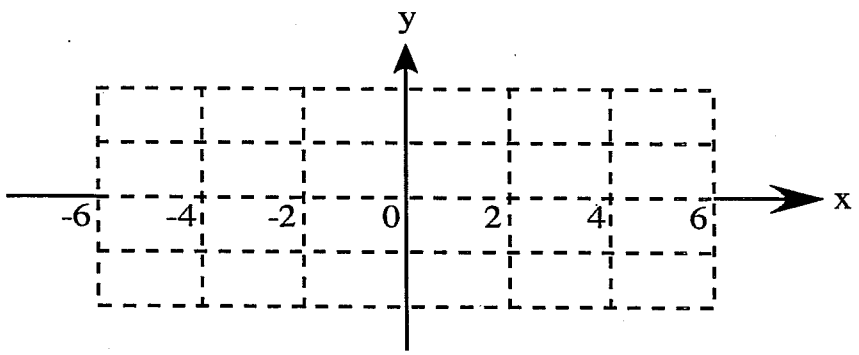
(a) Use the above recurrence relation to find a general solution of the above differential equation. Write any infinite series using the *sigma* notation and *simplify* as much as possible.

(b) Determine a **guaranteed** open interval of convergence for *each* of your two linearly independent solutions. Briefly explain your reasoning.

VALUES

(12) 6. (a) Calculate the Fourier **Cosine** series of $f(x)$ on the interval $[0, 2]$, where $f(x) = (x - 1)$, $0 \le x \le 2$ and ***simplify*** your answer.

(b) Using the "graph paper" below, draw the graph of the Fourier Cosine series of $f(x)$, as found in part (a), for $-6 \le x \le 6$, and indicate clearly (on the graph) the scale on the y–axis and the value at each point of discontinuity:



VALUES

(29) 7. The equation for the transverse vibrations of a string of length 5 (m) with the speed of propagation of waves along the string equal to 2 (m/s) is given by:

$$4u_{xx} = u_{tt}, \quad 0 < x < 5, \quad t > 0,$$

where $u(x, t)$ is the displacement of the string at the point x, at time t.

The boundary conditions are given by:

$$u(0,t) = u(5,t) = 0, \text{ for } t > 0.$$

The string's initial displacement is given by: $u(x,0) = 0$,

and its initial velocity is given by: $u_t(x,0) = 7$, for $0 < x < 5$.

(a) Solve the problem for $u(x, t)$ by using separation of variables.
Show all the details of your calculations and simplify your answer.

(b) What is the physical interpretation of the boundary conditions?

(c) Is the solution periodic in time? If so, what is its period?

Note: there is extra space on the next page to continue your work. ⇨

EXTRA SPACE for # 7:

(10) 8. **This question asks you ONLY to fill in the table below.**

A thin, laterally insulated rod has diffusivity 3 and length 10.

Let $u(x, t)$ be the temperature (in degrees Celsius) of the rod at the point x, at time t.

The left end of the rod is insulated.

The right end of the rod radiates heat into the surrounding medium; this medium has a temperature of (-10) degrees Celsius.

The initial temperature of the rod is given by a parabolic function (i.e., a quadratic polynomial) which is zero at both end-points and has a maximum value of 25 degrees Celsius (at the center point).

Write down the PDE, the boundary conditions, and the initial condition(s) that the temperature, $u(x, t)$, must satisfy; include appropriate bounds on the variables; also include, as appropriate, any constraints on parameters that appear in your equations.

Do NOT solve the above problem for $u(x, t)$.

	EQUATION	BOUND(S) ON VARIABLE(S)
PDE:		
Boundary Condition Left end point:		
Boundary Condition Right end point:		
Initial Condition(s):		

* *

Total = 100

References

***Formulae* from the Theorems of Vector Calculus:**

Green's Theorem: $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

Divergence Theorem: $\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV$.

Stokes's Theorem: $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS$.

See SUMMARY OF FOURIER SERIES on next page ⇨

SUMMARY OF FOURIER SERIES

I. FULL SERIES: Let f(x) be defined and piece-wise smooth on the interval $\{x : a \le x \le b\}$. $(a < b)$.

Let $p = (b-a)/2$; the length of the interval is $b - a = 2p > 0$.

The Fourier series of $f(x)$ on the interval $[a, b]$ has a period of $2p = b - a$ and is given by:

$$f(x) \sim A + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right], \quad a < x < b.$$

$$A = \frac{1}{2p}\int_a^b f(x)\,dx, \qquad a_n = \frac{1}{p}\int_a^b f(x)\cos\left(\frac{n\pi x}{p}\right)dx, \qquad b_n = \frac{1}{p}\int_a^b f(x)\sin\left(\frac{n\pi x}{p}\right)dx.$$

The series converges to $\left\{\begin{array}{ll} \frac{1}{2}\left[f(x^+) + f(x^-)\right], & a < x < b \\ \frac{1}{2}\left[f(a^+) + f(b^-)\right], & x = a, b \end{array}\right\}$.

II. HALF-RANGE EXPANSIONS: Let $f(x)$ be piece-wise smooth on $\{x : 0 \le x \le L\}$. $(L > 0)$.

(a) FOURIER COSINE SERIES: Extend f to an *even* function for $-L \le x \le L$.

The Fourier Cosine series has a period of $2L$; it equals $f(0^+)$ at $x = 0$, and $f(L^-)$ at $x = L$.

$$f(x) \sim A + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad 0 < x < L; \qquad A = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$$

(b) FOURIER SINE SERIES: Extend f to an *odd* function for $-L \le x \le L$.

The Fourier Sine series has a period of $2L$; it equals 0 at $x = 0$ and L.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 < x < L; \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

III. FORMULAE: (k is a non-zero constant): $\int \cos kx\,dx = \frac{1}{k}\sin kx$, $\int \sin kx\,dx = -\frac{1}{k}\cos kx$,

$$\int x\cos kx\,dx = \frac{1}{k^2}\cos kx + \frac{x}{k}\sin kx, \qquad \int x\sin kx\,dx = \frac{1}{k^2}\sin kx - \frac{x}{k}\cos kx,$$

$$\int x^2\cos kx\,dx = \frac{x^2}{k}\sin kx + \frac{2x}{k^2}\cos kx - \frac{2}{k^3}\sin kx,$$

$$\int x^2\sin kx\,dx = -\frac{x^2}{k}\cos kx + \frac{2x}{k^2}\sin kx + \frac{2}{k^3}\cos kx.$$

For any integer n:

$$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n, \quad \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n, \quad \cos\left(\frac{(2n+1)\pi}{2}\right) = 0.$$

END OF EXAMINATION

Have a happy and safe holiday !!!