University of Manitoba
Department of Mathematics

Ph.D. Comprehensive Examinations
REGULATIONS and SYLLABI

Date: October 13, 2022

Ph.D. Comprehensive Examinations

A. Regulations

1. From the Department of Mathematics Supplemental Regulations for the Ph.D. Program: "The candidacy examination in the Ph.D. program in Mathematics consists of a single written comprehensive examination chosen from the following areas:

   - Algebra;
   - Analysis.

   Regulations governing these examinations and the latest syllabi on which these examinations are based are described in the document "Ph.D. Comprehensive Examinations REGULATIONS and SYLLABI" approved by the Department Council, and are available from the Associate Head (Graduate Studies).

   The Graduate Studies Committee arranges comprehensive examinations two times a year, normally between the 15th and 25th days of April and September. The student must register to write a comprehensive examination by sending a request to the Associate Head (Graduate Studies) by February 1 if the examination is given in April and by July 1 if the examination is given in September. In some exceptional cases, the Graduate Studies Committee may allow a student to register after these deadlines upon receiving a formal letter of request from the student detailing their reasons for the late registration. The choice of area and specialized topics must be approved by the student's advisor for each exam.

   The standard of pass shall be given on the question sheet of each examination. A student who fails a comprehensive examination in any area twice shall be required to withdraw from the Ph.D. program in Mathematics."

2. Once a student has made a formal request for an examination (either in writing or by email), he/she is obligated to write it (except that a request can be withdrawn before March 1 if the examination is given in April and before August 1 if the examination is given in September). Absence from the examination on medical or compassionate grounds will be excused according to the same policies that apply to final examinations in the Faculty of Science. Any other delay or deferral of the examination shall be considered by the Graduate Studies Committee only upon receipt of a written request from a student's advisor (this request must be received by the Graduate Studies Committee before the examination) outlining the specific reasons for the delay or deferral.

3. Each Examining Committee shall consist of at least three persons. One member of each committee will be designated as Coordinator and will be responsible for communicating with the students, with the Graduate Studies Committee through the Associate Head (Graduate Studies) and with the Department office (through the Administrative Assistant).

   The Examining Committee sets the questions for the examination as it sees fit. Once the examination has reached its final form it is the responsibility of every committee member to read the entire examination and ensure that it is consistent with the syllabus and with the general level of difficulty expected.

   The Coordinator is responsible for ensuring that all the regulations in this document are adhered to.

4. **This regulation applies to all comprehensive examinations unless the syllabus for an examination explicitly overrides it.**

Style of the examination:

The examination shall consist of up to three parts A, B and C.

a) Part A shall consist of N_1 mandatory questions, worth a total of W_1 marks (individual marks for each question may or may not be the same). Here, N_1 is a positive integer and W_1 is positive.
b) Part B (if any) shall provide a choice of N_2 questions worth ω_2 marks each with the instruction that the student is allowed to attempt M_2 out of N_2 questions (M_2 shall be strictly less than N_2) with a clear indication of which M_2 questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_2 := \omega_2 M_2$).
c) Part C (if any) shall provide a choice of N_3 questions worth ω_3 marks each with the instruction that the student is allowed to attempt M_3 out of N_3 questions (M_3 shall be strictly less than N_3) with a clear indication of which M_3 questions he/she wants to have marked (and so the maximum marks that a student can obtain in this part shall be $W_3 := \omega_3 M_3$).

It is up to the Examining Committee to decide how many questions will be in each part and how many marks will each part be worth (i.e., there are no restrictions on the values of N_i, M_i and W_i), and there is no obligation to follow the format/style of past examinations. The core material (UNIT ONE) will comprise 40% of the total grade (i.e., $\frac{4}{10}(W_1 + W_2 + W_3)$), while the two specialized topics of UNIT TWO will each comprise 30% of the total grade (i.e., $\frac{3}{10}(W_1 + W_2 + W_3)$). However, students shall be given 6 hours to complete each examination. This time limit is intentionally generous. The expectation is that the examination can be successfully finished in less time (sometimes significantly less). Students shall be given all the questions of the entire examination at the beginning of the examination.

Passing criteria: In order to pass the examination the student needs to obtain at least 75% overall, i.e., the minimum passing mark shall be $\frac{3}{4}(W_1 + W_2 + W_3)$.

5. The style of the examination (i.e., the number of parts and their description explicitly mentioning the values for N_i, M_i and W_i), and the explicit passing criteria shall be provided on the examination paper (in a manner understandable to a student). The Examining Committee may provide additional instructions, recommendations, expectations and/or any other information on the examination paper as well.

   All the information described above in this section shall be communicated to student(s) 1 (one) working day before the examination date. (For example, the first page of the examination can contain this information and be set up by the Examining Committee so that it could be photocopied and given to student(s).)

6. The Examining Committee assigns the task of detailed grading of the various problems on each examination paper among the Committee members as it sees fit.

   In particular, individual examiners may be assigned particular questions to grade; or every examiner may be asked to grade the entire examination in detail. However, regardless of the particular methods used, in the end it is the responsibility of every Committee member to read the entire paper, and to form at least a general opinion of the performance of the student.

7. No member of the Examining Committee shall write on a student's original examination paper. Copies of the paper shall be made for each examiner's personal use.

8. After all the papers have been graded the Examining Committee shall have a meeting (virtual meeting or "meeting by email" is allowed) to decide on the marks to be given for each question

on each of the examination papers. The Examining Committee decides how these marks are to be given as it sees fit (for example, by rounding the averages of the marks given by each examiner, by re-marking each problem in a committee, etc.). Once this marking is complete, the Examining Committee shall use the "pass criteria" already set and indicated on the examination paper to determine if the student passes or fails the examination. The Coordinator shall write a report containing the mark decided upon by the Examining Committee, but not marks by individual examiners, of each question and comments (if any). The report must clearly indicate the result of the examination ("pass" or "fail"). It cannot contain statements of the sort "passed/failed notwithstanding". The report shall be given to the Associate Head (Graduate Studies) who will forward it the candidate as well as the Department for storage.

9. During the appeal period, neither the Coordinator nor the Committee shall communicate with the students, their advisors or anybody else other than the Associate Head (Graduate Studies) and the Department Head on any matter pertaining to the examination unless the Associate Head or the Department Head instructs otherwise (in writing).
10. The Associate Head (Graduate Studies) or Department Head shall announce the results formally to each of the students by means of a letter on Department letterhead. A copy of each letter must be provided to the Department office for the student's files. In the case of a failure, the letter shall inform the student of the right to appeal the outcome (with a reminder about the internal two week deadline). In the case of a first failure, the letter shall inform the student of the possibility to rewrite the examination; and in the case of a second failure on a given examination that this result means that the student will be required by the Faculty of Graduate Studies to withdraw from the Ph.D. program.
11. In the case that a first failure has been reported, and after the appeal period is over, it is the responsibility of the Coordinator to ensure that the student receives general feedback on the reasons for the failure and some guidance in study and preparation for the second attempt.
12. After the result of the examination has been announced, the student may view a copy of his/her examination paper in the Department office, under supervision. Students shall not be allowed to make any copies of any of the examination papers at any time.
13. **Appeals.** If a student wishes to appeal the result of his/her examination, a formal appeal in writing must be addressed to and received by the Department Head within 10 (ten) working days of the announcement of the result to the student. Appeals based solely on disagreement of the allocation of marks for one or more questions will be automatically rejected.

B. Advice to students preparing for Ph.D. Comprehensive Examinations

1. A Graduate Comprehensive Examination in the Department of Mathematics is an examination of material that is normally taught in an undergraduate honours program. For greater clarity, any material that is covered by the syllabus may appear on a comprehensive examination irrespectively of whether or not it has been (or is) taught in undergraduate or graduate courses at the University of Manitoba or any other institution.

   Additionally, since it is impractical (if not impossible) to list all possible definitions/concepts in the syllabus, it should be understood that topics related to those in the syllabus are also covered. For example, if a section of a book is covered by a syllabus, and a topic is discussed in this section, then this topic may be covered by the examination even if that specific concept is not explicitly mentioned in the syllabus. It goes without saying that you should also know all the basic material that is a prerequisite for the topics that are covered by the syllabus.

Even though comprehensive examinations are based on undergraduate material, they are being written by Ph.D. students, and so a more sophisticated level of understanding and presentation may be expected than might normally be demanded in relevant undergraduate courses.

2. Copies of old comprehensive examinations may be available from the Department office or from the Department website. However, you should have ABSOLUTELY NO EXPECTATIONS that future examinations will be identical or even remotely similar to any particular previous examination. Do not base your expectations for the examination on any particular previous examination.
3. Do not limit your study and preparation to one reference; reviewing all references provided with the syllabus will provide a much better preparation for different styles of questions.
4. Texts and old examinations from relevant undergraduate courses may be useful as a study supplement, but students should master the material in the syllabus of the comprehensive examinations which follow.
5. Do not expect any Faculty member to "teach" you the subject matter of the examination (outside the regular coursework). You are admitted to our Ph.D. program on the basis that you are (mostly) prepared for work at this level, including being prepared (or having the mathematical maturity to be able to prepare) for the Comprehensive Examinations. You should be able to prepare for these examinations by reviewing of your earlier studies, and by self-studying of any subjects that you have missed.
6. Faculty members may be able to advise you on relevant study plans, be helpful with specific difficulties that you may have, and so on. The Associate Head (Graduate Studies) can help clarify the regulations; the Coordinator of the Examining Committee may help you interpret the details of the Examination Syllabus; and members of Faculty working in the general area of the Examination are often available for help with specific questions.
7. Expect a mixture of theoretical and practical problems, sometimes in the same question. Classifying questions as theoretical or practical is in many cases arbitrary, and a comprehensive examination in any case may (and often will) cover the entire range of styles of questions suitable to the subject area.
8. Corrective action can be taken at any time if mistakes in grading and/or questions are found, even after expiration of the appeal period.

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Algebra Comprehensive Examination Syllabus

The algebra comprehensive examination consists of two units: UNIT ONE, which is for the core material and UNIT TWO, which is for the specialized topics. For UNIT TWO, each student shall choose 2 of the 6 subsections II.(a–f) below that will be covered by the examination. A student cannot choose as their specialized topics II.(b) and II.(c). The student must declare his/her choices when registering for the examination. UNIT TWO of the examination will only cover the two subsections that the student has previously chosen. If a topic is mentioned in both the core material and in a specialized subsection, it is to be understood that a greater depth of knowledge of the topic is expected if the student chooses to write that specialized subsection.

I. **The core material**

Students are expected to know basic set theory and number theory, including topics such as functions, equivalence relations, permutations; divisibility, primes, congruences modulo n, the Euclidean Algorithm, GCD, LCM, and prime power decompositions of integers. In addition, the following topics are required:

- ▷ Groups: Definition of a group, subgroups and subgroup lattices, cyclic groups, definition and basic examples of abelian groups, permutations groups, homomorphisms, isomorphisms, cosets, normal subgroups, modular lattices, factor groups, Lagrange's Theorem, isomorphism theorems, Sylow Theorems, abelian groups, solvable groups, simple groups, group actions on sets, groups acting on themselves (by translation, by conjugation), orbit-stabilizer theorem, finite direct products of groups, finite direct sum of abelian groups, fundamental theorem of finite abelian groups.
- ▷ Rings: Commutative rings: the ring of integers, polynomial rings, integral domains. Non-commutative rings: rings of linear operators on a vector space, rings of matrices, the ring of endomorphisms of an abelian group. Ideals and factor rings, prime ideals, maximal ideals. Polynomial rings: irreducible polynomials, factorizations, Eisenstein criterion, principal ideal domains, euclidean domains, unique factorization domains.
- ▷ Fields: Basic examples, such as the fields of rational, real, and complex numbers, fields of quotients of integral domains, fields as homomorphic images of commutative rings, basics of finite fields. Field extensions: finite, algebraic, and transcendental extensions. Finite fields and splitting fields.
- ▷ Vector spaces and linear algebra: Examples of vector spaces: polynomials, $\mathbb{C}^n$, matrices, linear transformations and operators, function spaces. Abstract vector spaces: definitions and first properties, subspaces, linear independence, spanning, bases, dimension. Coordinatization of finite dimensional vector spaces. Direct sums. Dimension theorems. Matrix representations of linear transformations and operators of an abstract vector space. Change of basis. Inner product spaces: length of and angles between vectors. Gram-Schmidt orthogonalization. Orthogonal projections and approximations. Diagonalization of matrices: real, symmetric matrices. Jordan form. Normal, orthogonal and Hermitian matrices.

Suggested reading list for core material:

- ▷ Primary references:

[1] P. M. Cohn, *Classic Algebra*. [QA 154.2 C63 2000].
[Chapters 1–9, 10.1–10.5, 10.8, 10.11, 11.1–11.5, Appendix 1]

[2] P. M. Cohn, *Basic Algebra: Groups, Rings, and Fields*. [QA 154.3 C64 2003].
[Sections 1.1, 1.2, 2.1–2.5, 4.1, 4.2, 4.4–4.8, 7]
The rest of this book is well beyond the scope of this syllabus.

▷ For an in-depth review of Linear Algebra and Galois Theory:

[3] K. Hoffman and R. Kunze, *Linear Algebra*. [QA 252 H67 1971].

[4] I. Stewart, *Galois Theory*. [QA 214 S74 2004].

▷ Additional reading:

[5] M. Artin, *Algebra*, 2nd ed. [QA 154.2 A77 2011].

[6] J. B. Fraleigh, *A First Course in Abstract Algebra*, 3rd ed. [QA 162 F7 1982].

[7] I. N. Herstein, *Topics in Algebra*. [QA 155 H4 1954].
(**Not** Abstract Algebra, which is more elementary)

[8] N. Jacobson, *Basic Algebra, Vol. I*. [QA 154.2 J32 1985 V.1].

[9] S. Lang, *Algebra*. [QA 154 L35 1965].
(**Not** Undergraduate Algebra, which is more elementary)

II. **The specialized topics**

(a) **Advanced Group Theory**

- ▷ Infinite direct products of groups, infinite direct sum of abelian groups, semidirect products;
- ▷ free groups, universal property of free groups, presentations, generators and relations;
- ▷ fundamental theorem of finitely generated abelian groups;
- ▷ p-groups, subnormal and composition series, solvable groups, nilpotent groups, ascending central series, derived series, descending central series, Jordan-Hölder Theorem;
- ▷ elementary definitions: representation of a group, subrepresentation, (iso)morphism of representations, irreducible (simple) representation, completely reducible (semisimple) representation, decomposable representation, unitary representation;
- ▷ Maschke's Theorem, Schur's Lemma, Weyl's 'averaging trick';
- ▷ new representations from old: dual, direct sum, tensor product, Hom-spaces, isotypical decompositions, isotypical summands, alternating and symmetric product;
- ▷ character theory for finite groups: character of a representation, class function, orthogonality of irreducible characters, character table, orthogonality relations, properties of characters;
- ▷ algebraic integers, Dimension Theorem (dimension of irreducible divides order of group);
- ▷ the regular representation, the group algebra, the Fourier transform (i.e., expressing $\mathbb{C}[G]$ as a direct sum of endomorphism algebras of irreducible representations of G)
- ▷ restriction, induction, Frobenius reciprocity, induced characters.

The topics listed above in this subsection may be found in Hungerford's Algebra (Chapters 1 and 2), Representation Theory of Finite Groups by B. Steinberg (Chapters 3 to 6, and 8), and Representation Theory: a First Course, by Fulton and Harris (Sections 1 to 4).

(b) **Commutative Algebra**

- ▷ Module theory: definitions, examples, quotient modules, module homomorphisms, generation of modules, direct sums of modules, free modules, tensor products of modules, modules over PIDs (Principle Ideal Domains), localization and local properties, Nakayama's Lemma
- ▷ Commutative Rings: definitions, examples, extension, contraction, radical of an ideal, Jacobson radical, nilradical, primary decomposition of an ideal, associated primes, localization, discrete valuation rings, integral closure, Krull dimension, height
- ▷ Finiteness Conditions: Noetherian rings, Artinian rings, Dedekind Domains
- ▷ Commutative Local Rings: characterization with units (Proposition 45 of Dummit and Foote's or Proposition 1.6 of Atiyah and Macdonald's references)
- ▷ Homological Algebra: definitions, examples, chain complexes, exact sequences - projective, injective and flat modules, "diagram chasing" (Snake Lemma, 5-Lemma, 9-Lemma, Horseshoe Lemma), Ext and Tor, cohomology of groups

References:

- ▷ Abstract Algebra, 3rd edition, by David S. Dummit and Richard M. Foote (Chapter 10, Sections 12.1, 15.1–15.4 (not including variety-specific material), Chapter 16, Sections 17.1, 17.2)
- ▷ Introduction to Commutative Algebra, by M. F. Atiyah and I. G. Macdonald (Chapters 1–9)
- ▷ Commutative Algebra with a View Towards Algebraic Geometry by D. Eisenbud (Appendix 3 for statements of "diagram chasing" lemmas)
- ▷ An Introduction to Homological Algebra by C. Weibel (Lemma 2.2.8 for statement of Horseshoe Lemma)

(c) **Non-commutative Algebra**

- ▷ Basics of Module Theory. ([DF, 10.1–10.3], [Ro, Ch 0; 1.4] [St, I.1, I.2; I.4; III.1–III.4], [Da, Ch 1]
  Modules over a ring. Submodules, homomorphisms, diagrams, exact sequences. The lattice of submodules and basic lattice-theoretic concepts (modularity, distributivity, complete and algebraic lattices)
  The Hom group, the endomorphism ring, bimodules.
  Direct sums and products.
- ▷ Standard constructions of rings [DF, 7.2], [Ro, 1.1, 1.2], [St, I.3], [GW, Prologue, Ch 1, Ch 2], [MR, Ch 1]
  Noetherian and artinian modules and rings.
  Matrix rings, (skew) polynomial rings, group rings and related constructions.
- ▷ Ideals of rings. [DF, 7.2, 7.4], [Ro, Ch 0, 2.1, 2.2], [GW, Ch 3], [Da, Ch 7]
  Left and right ideals.
  Prime ideals, semiprime ideals, annihilators, associated primes.
  Primitive and semi-primitive ideals and the Jacobson Radical.
- ▷ Free modules and generators. [DW, 10.3], [Ro, 0.2, 1.3] [St, I.2, I.3],[Da, Ch 2]
  Free modules, basis.
  Finitely generated and finitely presented modules.
- ▷ The category of modules. [DF, Appendix 2, 10.5], [Ro, 0.1, 1.4, 1.8, but <u>not</u> Ch 4], [St, IV 1-6, 8], [Da, Ch 14, 15 1,2, Ch 16]
  Categories and functors. Duality. Natural transformations. Abelian categories.
  Categorical concepts as exemplified in categories of modules:
  Basic constructions: product, coproduct, pushout, pullback. Kernel, cokernel, image.
  Initial and terminal objects. Generators and co-generators.
  (left and right) exact functors.
  Limits [projective, inverse limits] and colimits [inductive, direct limits] and their construction in the category of modules.
  Elements of homological algebra.
- ▷ Tensor product of modules; tensor algebras. [DF, 10.4, 11.5], [Ro, 1.7], [St, I.8,9], [Da, Ch 4, Ch 5]
  The definition, existence, and construction of tensor products. Tests for a zero product.
  Tensor as a bifunctor.
  The Hom-Tensor duality, with attention to the bimodule structure.
  Algebras over a field, graded algebras.
  The construction of (graded) tensor algebras.
- ▷ Semisimple modules and rings. [Ro, 2.3, 2.4], [St, I.7], [GW, Ch 4], [Da, 7.1, 10.1]
  Definition and basic properties.
  Artinian modules. Artinian rings are noetherian.
  Artin-Wedderburn Theorem (also known as Wedderburn-Artin Theorem or Wedderburn's Theorem).
- ▷ Projective modules. [DF, 10.5], [Ro, 2.8], [St, I.6], [Da, 10.1, 13.1, 13.2]
  Projective modules. Dual basis theorem.
  Hereditary and semi-hereditary rings, hereditary artinian rings.
- ▷ Injective modules. [DF, 10.5], [Ro, 2.10], [St, I.6], [GW, Ch 5], [Da, Ch 3]

Injective modules. Essential extensions, injective hull.
Uniform modules, indecomposable injective modules.
Injective modules over (left or right) noetherian rings.

References and suggested reading:

DF David S. Dummit and Richard M. Foote, **Abstract Algebra**, 3rd edition. A standard comprehensive early graduate level textbook on all aspects of abstract algebra, and has good sections on some of the topics of this syllabus. The emphasis is mostly on the commutative context.
Primarily material in Chapters 7 and 10.

Ro Louis Rowen, **Ring Theory, Vol. 1**, An older comprehensive text on ring theory, with good coverage in Volume 1 of the topics of this Syllabus.
Primarily material in Chapter 0, Chapter 1, and parts of Chapter 2.

St Bo Stenström, **Rings of Quotients**. This is not a text on the topics of this exam, but the introductory chapters contain concise and detailed expositions of many of the topics listed.
Primarily material in Chapter I, 1–9, Chapter III, 1–4, and Chapter IV, 1–8.

GW K. Goodearl and R. Warfield, Jr, **An introduction to non-commutative noetherian rings**. The standard introductory text to non-commutative ring theory at the graduate level.
Primarily the Prologue and Chapters 1–5.

MR J. C. McConnell and J. C. Robson, **Non-commutative noetherian rings**. A more advanced and detailed treatment than [GW], with excellent examples.
Chapters 0 and 1 only.

La T. Y. Lam, **Lectures on Modules and Rings**. Good coverage of projective and injective modules

Da John Dauns, **Modules and Rings** A comprehensive text covering a variety of topics in this syllabus. A good source of exercises.
Primarily Chapters 1–5, 14–16 and parts of Chapters 7, 9, 10, and 13.

(d) **Combinatorics**

Principles of counting and relations

Basic counting: Counting rules (product and sum rules), permutations, r-permutations, subsets, r-combinations, binomial coefficients, probability, sampling with replacement, occupancy problems (e.g., indistinguishable balls into distinguishable cells), Stirling numbers of the second kind, multinomial coefficients, binomial expansion, generating permutations and combinations, Gray codes, algorithms and complexity, pigeonhole principle, partitions of integers and Ferrers diagrams.

Relations: Binary relations, order (linear, partial), lexicographic order, linear extensions, chains, antichains, Sperner's lemma, Dilworth's theorem, interval orders, lattices, distributive and modular lattices, complementation in lattices, Boolean lattices.

Generating functions: Applications to distribution, composition, and partitions of integers, generalized binomial theorem, exponential generating functions, counting permutations, distinguishable balls into indistinguishable cells.

Recurrence relations: Fibonacci numbers, derangements, method of characteristic roots, recurrences involving convolutions, Catalan numbers (including counting lattice paths, triangulations), solution by generating functions.

Inclusion and exclusion: Principle of inclusion-exclusion, counting derangements, number of objects having exactly m properties.

References for counting and relations

[RoTe:05] F. S. Roberts and B. Tesman, Applied combinatorics, 2nd ed., Pearson Prentice Hall, Upper Saddle River, NJ, 2005. [QA164.R6.2005] See Chapters 2, Sections 4.1, 4.2, 4.3.2, Chapters 5–7.

[Brua:10] R. A. Brualdi, Introductory combinatorics, 5th ed., Pearson Prentice Hall, 2010. [QA164.B76.2010] See Sections 2.1–2.5, Sections 3.1, 3.2, 4.3, 4.5, 5.2, 5.3, 5.6, 6.1, 6.6, 7.1–7.4, 8.1, 8.3.

[vLWi:05] J. H. van Lint and R. M. Wilson, A course in combinatorics, 2nd ed., Cambridge University Press, Cambridge, UK, 2001. [QA164.L56.2001] See Chapters 6, 8.

[Came:94] P. J. Cameron, Combinatorics: topics, techniques, and algorithms, Cambridge University Press, Cambridge, UK, 1994 (with corrections, 1996). [QA164.C346.1994] See Sections 12.1–12.3, 20.1.

Graph theory

Basic graph theory: Graph, multigraph, labelled graph, degree, degree sequence, handshaking lemma, Havel–Hakimi theorem, isomorphism, subgraph, complete graph, regular graph, bipartite graph, complement, hypercubes, connected, cut vertex, block, bridge, distance, walk, adjacency matrix, counting walks, incidence matrix, trail, path, Dijkstra's algorithm, circuit, cycle, girth, radius, diameter, centre, eccentricity.

Trees: Their properties, chemical bonds and trees, Cayley's theorem for labelled trees, Prüfer sequences, minimum spanning trees, Kruskal's algorithm, Prim's algorithm.

Directed graphs: Digraph, tournament, strongly connected, transitive tournament, king, Robbin's theorem for one-way streets,

Connectivity: Vertex and edge connectivity, Menger's theorem (graph vertex version), Whitney's theorem.

Cycles and circuits: Eulerian circuits and trails in graphs and digraphs, Fleury's algorithm, DeBruijn sequences, rotating drum problem, Chinese postman problem, Hamiltonian paths and cycles, Travelling Salesman Problem, Ore's condition, Dirac's condition, independent set.

Planar graphs: Euler's formula for planar graphs, five regular polyhedra, planar duals, Four Colour Theorem (statement only), Kuratowski's and Wagner's theorems (statements only).

Colouring: Vertex colourings, chromatic number, Brooks' theorem, Nordhaus–Gaddum theorem, perfect graph, interval graph, chordal graph, edge colouring, chromatic index, Vizing's theorem (statement only), König's theorem (for chromatic index), Ramsey's theorem, small Ramsey numbers, upper and lower bounds for $R(k, k)$.

Matchings, covers, and flows: Flows in networks, max-cut min-flow algorithm, matchings, augmenting paths, perfect matchings, 1-factor, Hall's marriage theorem, systems of distinct representatives, stable marriages, Gale–Shapley algorithm, factorizations, Tutte's 1-factor theorem, Petersen's theorem for cubic bridgeless graphs, vertex covers, edge covers, Gallai's theorem (for independence and covering numbers), König–Egerváry theorem.

References for graph theory

[CLZ:11] G. Chartrand, L. Lesniak, and P. Zhang, Graphs and digraphs, 5th ed., Chapman and Hall/CRC, 2011. See Chapters 1, 2, Section 3.1, 3.2, Chapter 4, Sections 6.1, 6.2, 8.1–8.4, 9.1, 10.1–10.3, 11.1, 11.2, 12.3.

[RoTe:05] See Sections 3.1–3.3.4, 3.5, 3.7, 3.8, 11.2.1, 11.3, 11.4.1, 11.5, 11.6.1, 12.1, 12.2, 12.4, 12.5, 12.8.1. 13.1, 13.2, 13.3, 13.4.

[Brua:10] See Sections 3.3, 11.1–11.5, 11.7, 12.1–12.3, 12.5, 12.6, Chapter 13.

Elementary design theory

BIBDs, dual and complement of a design, latin squares, problem of 36 officers, MOLS, latin squares over finite fields, finite projective planes, homogeneous coordinates, incidence matrices, t-designs, derived and residual designs, Hadamard matrices, resolvable designs, Kirkman's school girl problem, Steiner triple systems, Fisher's inequality, and statements of Bruck-Ryser-Chowla and Bruck-Ryser theorems.

References for design theory

[RoTe:05] See Sections 9.2.2, 9.2.3, 9.3, 9.4.1–9.4.7, 9.5.

[Brua:10] See Sections 1.4, Chapter 10.

[vLWi:05] See Chapter 19.

[Came:94] See Chapter 8.

(e) **Galois Theory**

- ▷ Finite fields, uniqueness of finite fields of a given order, orders of finite fields, Frobenius automorphism. Prime subfields.
- ▷ Field extensions. Finite, algebraic, and finitely generated extensions. Primitive elements. Minimal polynomial. Roots of polynomials. Adding a root of a polynomial. Transcendental extensions and transcendance degree.
- ▷ Quadratic extensions, straightedge and compass constructions, constructible numbers.
- ▷ Splitting fields. Roots of unity. Cyclotomic extensions and cyclotomic polynomials. Algebraically closed fields and their construction.
- ▷ Automorphism groups of field extensions. Separable and normal extensions. Subfield fixed by a group of automorphisms. Galois extensions and the Galois group of an extension. Fundamental Theorem of Galois Theory.
- ▷ Determining the Galois group of a polynomial, discriminants. Galois group of the splitting field of a polynomial and permutations of the roots of the polynomial.
- ▷ Extensions by radicals. Equations "solvable by radicals", basics of solvable groups. The general equation of degree n and its Galois group. Elementary symmetric polynomials and fundamental theorem of symmetric polynomials.
- ▷ Primitive element theorem, normal basis theorem, independence of characters, Hilbert's Theorem 90.

Suggested reading list:

[10] T. Hungerford, *Algebra*. Chapters 5 and 6.

[11] I. Stewart, *Galois Theory*.

[12] J.S. Milne, *Fields and Galois Theory*. Chapters 1–6.

(f) **Linear Algebra**

- Infinite dimensional vector spaces. Direct sums. Every vector space has a basis (application of Zorn's Lemma or equivalent).
- Linear operators on finite and infinite dimensional spaces.
- Linear functionals. Duals of both finite and infinite dimensional spaces.
- Eigenvalues and eigenvectors of a linear operator; of a matrix. Characteristic equation of a matrix. The minimal polynomial. Diagonalization of matrices (operators) and applications. Invariant subspaces. Cayley-Hamilton theorem. Jordan canonical form.
- Bilinear forms and their representations. Quadratic forms. Positive definite and positive semidefinite bilinear forms.
- QR factorization
- Schur triangularization and applications
- Singular value decomposition
- Matrix norms: vector norms and inner products, matrix norms
- Nonnegative matrices: irreducible and primitive matrices, Perron-Frobenius theorem.

Suggested reading list:

[1] P. M. Cohn, *Classic Algebra.* [QA 154.2 C63 2000].

[2] D. Dummit and R. Foote, *Abstract Algebra.* [QA 162.D85 2004], Chapter 11, including exercises.

[3] K. Hoffman and R. Kunze, *Linear Algebra.* [QA 252 H67 1971].

[4] R.A. Horn and C.R. Johnson, *Matrix Analysis.* [QA188.H66 2013].

Analysis Comprehensive Examination Syllabus

The analysis comprehensive examination consists of two units: UNIT ONE, which is for the core material and UNIT TWO, which is for the specialized material. For UNIT TWO, each student shall choose 2 of the 6 subsections II.(a–f) below that will be covered by the examination. The student must declare his/her choices when registering for the examination. UNIT TWO of the examination will only cover the two subsections that the student has previously chosen.

I. **The core material**

Students are expected to know all basic mathematical analysis/calculus (see, e.g., [6, Chapters 1-9] and corresponding sections from [5]). In addition, the following topics are required:

- Functions of bounded variation: monotonic functions, total variation, functions of bounded variation expressed as the difference of increasing functions.
- Riemann and Riemann-Stieltjes integral: definition and elementary properties, integration by parts, change of variable, step functions as integrators, reduction of Riemann-Stieltjes integral to a finite sum, fundamental theorems of integral calculus.
- Functions: continuity and uniform continuity, extreme and intermediate value theorems, derivatives, mean value theorem.
- Sequences and series of functions: pointwise and uniform convergence, uniform convergence and continuity, Cauchy condition for uniform convergence, uniform convergence of infinite series of functions, radius of convergence of power series, uniform convergence and Riemann-Stieltjes integration, uniform convergence of improper integrals, uniform convergence and differentiation, sufficient conditions for uniform convergence of a series (Weierstrass M-test, Dirichlet's test), Weierstrass approximation theorem, Arzelà–Ascoli Theorem.
- Inverse function and implicit function theorems, theorems of Green, Gauss and Stokes, conservative vector fields.
- Basics of Hilbert spaces: inner product, triangle and Cauchy-Schwarz inequalities, orthonormal bases and sequences, orthogonalization, Bessel's inequality, Pareseval's identity, Fourier series of a function relative to an orthonormal system, theorem on best approximations, properties of the Fourier coefficients, orthogonal systems of functions in L^2 of an interval, Cauchy sequences and completeness, Riesz-Fischer theorem.
- Fourier Series: pointwise and L^2 convergence of trigonometric and exponential Fourier series, Dini's Test, Jordan's Test, partial and Cesàro sums, Fejér's Theorem.
- Lebesgue Integration

  Measurable functions, definition and properties of the Lebesgue integral, Fatou's Lemma, monotone convergence theorem, dominated convergence theorem, convergence in measure.
- Differentiation and integration

  Vitali's cover and Vitali's theorem, integral of the derivative of an increasing function, differentiation of the indefinite integral, absolutely continuous functions.
- Basic properties of complex numbers and analytic functions, elementary analytic functions, entire functions, Cauchy-Riemann Theorem, Inverse Function Theorem.
- Contour Integrals, Cauchy's Theorem, Cauchy's integral formula, Cauchy's inequalities, Liouville's Theorem, Fundamental Theorem of Algebra, Morera's Theorem, Maximum Modulus Theorem, Schwarz Lemma.

- ▷ Convergent series of analytic functions, Taylor series, Laurent series, zeroes of analytic functions and classification of singularities, Casorati-Weierstrass Theorem.
- ▷ Residue Theorem and evaluation of integrals by using the Residue Theorem.
- ▷ Identity Theorem, Rouche's Theorem and Principal of the Argument Theorem, Hurwitz's theorem, Open Mapping Theorem for analytic functions.

The topics listed above may be found in [5, Sections 6.1-6.7, 7.1-7.9, 7.19, 7.20, 9.1-9.11, 10.13, 11.1-11.6, 11.15], [8, Chapters 1-5 and Section 10.8] and [7, Chapters 1-6].

References:

[5] T. M. Apostol, *Mathematical analysis*, 2nd ed., Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974. [QA 300 A573 1974].

[6] R. C. Buck, *Advanced calculus*, 3rd ed., Waveland Pr. Inc., January 1, 2003. [QA 303 B917 2003].

[7] J. E. Marsden and Michael J. Hoffman, *Basic complex analysis*, 3rd ed., W. H. Freeman and Company, New York, 1999. [QA 331 M378 1999].

[8] H. L. Royden, *Real analysis*, 3rd ed., Macmillan Publishing Company, New York, 1988. [QA 331.5 R6 1988].

II. The specialized material

(a) Approximation Theory

- ▷ Density, existence, uniqueness

  Kolmogorov theorem. Chebyshev polynomials. Haar (Chebyshev) systems. Theorems of Weierstrass, Korovkin, Stone, Müntz, Mergelyan.

- ▷ Polynomial approximation

  Polynomial partition of unity. Moduli of continuity. Direct theorem for approximation by polynomials. Moduli of smoothness. Whitney inequality. Bernstein polynomials, shape preserving properties. Trigonometric polynomial kernels: Dirichlet, Fejer, Jackson, Stechkin. Stechkin's theorem (without proof). Bernstein inequality for trigonometric polynomials. Inverse theorem for approximation by trigonometric approximation. Bernstein inequality for algebraic polynomials. Markov inequality. Influence of endpoints in polynomial approximation. Inverse theorem for approximation by algebraic polynomials. Lipschitz spaces.

- ▷ Non-polynomial approximation

  Splines. Euler's ideal spline. Kolmogorov's inequality for derivatives. Spline bases. B-splines. Estimates on errors of approximation by splines. Widths. K-functionals and applications.

References:

The material covered in this section may be found in [10] and in Sections 1.1-1.4, 2.1, 2.6, 2.7, 2.9, 3.1-3.6, 4.1, 4.5-4.7, 5.1-5.3, 5.5-5.9, 6.1-6.4, 7.1-7.3, 7.6, 7.7, 8.1-8.6, 10.1-10.3 of [9].

[9] R.A. DeVore and G.G. Lorentz, *Constructive approximation*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 303, Springer-Verlag, Berlin, 1993. [QA 221 D44 1993].

[10] K. Kopotun, A. V. Prymak, and I. O. Shevchuk, *Introduction to Approximation Theory: Study Guide*, 2011.

(b) **Complex Analysis**

- ▷ Conformal mappings and linear fractional transformations,
- ▷ Spaces of analytic functions,
- ▷ Equicontinuity, Uniform convergence on compact sets, Completeness of families of analytic and meromorphic functions,
- ▷ Normal families, Montel's theorem,
- ▷ Riemann mapping theorem,
- ▷ Harmonic functions, Mean Value Property for harmonic functions,
- ▷ Maximum Modulus Theorems for harmonic functions,
- ▷ Solution of the Dirichlet problem,
- ▷ Harnack's inequality and Harnack's Theorem.

The topics listed above in this subsection may be found in [11, Chapters III, VII and X].

References:

[11] J. B. Conway, *Functions of one complex variable*, 2nd ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978. [QA 331 C659 1978].

(c) **Computational Mathematics**

I. **Introductory Numerical Analysis**

- Polynomial Interpolation: Lagrange interpolation, Divided differences, Natural and clamped cubic spline interpolation.
- Numerical differentiation and integration: Richardson's extrapolation, trapezoid and Simpson's rules, Newton-Cotes formulas, Composite numerical integration, Gaussian quadratures.
- Least squares approximation, orthogonal polynomials approximation (Legendre and Chebyshev), and trigonometric polynomial approximation.
- Direct methods for solving linear systems: Gaussian elimination, Pivoting methods, Matrix factorization and special types of matrices.
- Iterative methods for solving linear systems: Jacobi and Gauss-Seidel iterative methods, Relaxation methods, Error bounds. Power method for eigenvalue problems.
- Numerical solutions for nonlinear equations and nonlinear systems: Bisection method, fixed point method, Newton's method, secant method. Error analysis.
- Numerical solution for initial value problems for ordinary differential equations: Single step method (Euler's method, Runge-Kutta methods). Local truncation error. Consistency, stability and convergence of numerical methods.
- Numerical solution for boundary value problems of ordinary differential equations: Shooting methods for linear and nonlinear problems, finite difference methods for linear and nonlinear problems.

References:
The material covered in this section may be found in Chapters 1-11, 12.1-12.5, 13 of [1].

[1] E. Suli and D. Mayers, *An Introduction to Numerical Analysis*, Cambridge University Press, 2003.

II. **Numerical Analysis of PDEs**

- Explicit and implicit finite difference schemes for parabolic partial differential equations (PDEs) including Euler and Crank Nicholson schemes, convergence, consistency and stability (von Neumann stability, matrix stability).
- Finite difference schemes (second order, fourth order, upwind schemes) for elliptic PDEs, convergence, consistency and stability.
- Finite difference schemes for hyperbolic PDEs, convergence, consistency and stability, CFL condition.
- Finite element methods for elliptic PDEs: 1D and 2D finite elements, local and global interpolation error estimates, L^2 and energy norm error estimates, inverse estimates.
- Numerical Linear Algebra: Condition number, Krylov subspace methods, preconditioning.

References:
The material covered in this section may be found in Sections 2.1, 2.2, 2.4, 2.6, 2.7, 3.3, 4.1, 4.2, 4.4, 4.9.1, 8.4 of [2] and Chapters 1, 3.1-3.5, 4.3-4.5 and 6 of [3].

[2] C. Grossmann and H. G. Roos and M. Stynes, *Numerical Treatment of Partial Differential Equations*, Springer-Verlag, 2007.

[3] S.H. Lui, *Numerical Analysis of Partial Differential Equations*, Wiley-Interscience Publication, New York, 2011.

(d) **Differential Equations**

Unit I. Ordinary Differential Equations

- **General theory of ordinary differential equations**
  Existence, and existence and uniqueness theorems. Gronwall's lemma. Banach fixed point/ contraction mapping theorem. Continuation of solutions. Maximal solutions. Continuous dependence on initial conditions. Invariance of sets under flows. Picard's approximation. Basic techniques of resolution for scalar first-order equations (separable equations, integrating factors).
- **Linear systems of first-order differential equations**
  Transformation of a scalar n^{th}-order linear equation into a system of n scalar first-order differential equations. Homogeneous systems (superposition principle; the vector space of solutions – fundamental set of solutions; fundamental matrix). Autonomous homogeneous systems (diagonalizable case; generalized eigenspace; matrix exponential; trichotomy – stable, unstable and centre subspaces). Nonhomogeneous systems (variation of parameters). Laplace transform.
- **Planar systems**
  Direction fields. Nullclines. Phase plane analysis. Fixed points. Stability of fixed points. Classification of types of orbits/trajectories in the phase plane for systems with constant coefficients. Oscillations. Linearization of nonlinear systems: the stable manifold and Hartman-Grobman theorems. α- and ω-limit sets. Stability, asymptotic stability, Lyapunov stability.

Students are expected to know topics from analysis used in proofs in the suggested references; for example, complete metric spaces, Cauchy sequences, uniform continuity and uniform convergence, Lipschitz functions, etc. Tables of Laplace transforms will be provided, as needed.

Suggested references

[1] N.G. Markley, *Principles of Differential Equations*, Pure and Applied Mathematics (New York), Wiley-Interscience, Hoboken, NJ, 2004. [QA 371 M27 2004].

[2] L. Perko, *Differential Equations and Dynamical Systems*, Third edition, Texts in Applied Mathematics, vol. 7, Springer-Verlag, New York, 2001. [QA 372 P47 2001].

Markley [1] is the main reference for Unit I; all material in Chapters 1 to 5 is to be known, with the exception of Sections 3.4, 4.5 and 5.4. In Perko [2], use Chapter 1, Sections 2.1 to 2.12.

Unit II. Partial Differential Equations

- heat equation
- Laplace equation
- scalar conservation laws
- wave equation
- Lax–Milgram Theorem
- $H^1,\ H^1_0,\ H^{-1}$, compactness and embeddings
- Problems for the Poisson equation

Suggested references

[3] S. Salsa, *Partial differential equations in action. From modelling to theory*, 3rd Ed., Universitext, Springer-Verlag Italia, Milan, 2016.

Relevant material for Unit II is in contained in the following sections of [3]: 1.1 - 1.6, 2.1 - 2.3, 2.8, 3.1 - 3.3, 3.5 - 3.8, 4.1, 4.2, 4.4, 4.6.3, 5.1 - 5.4, 5.6, 6.6, 7.7.2 - 7.7.5, 7.10, 8.3. Note that this book is available online through the University of Manitoba Libraries.

(e) **Functional Analysis**

- ▷ L^p - Spaces

  Minkowski and Hölder inequalities, Completeness, Riesz representation theorem for the dual space of L^p. [4, Sections 7.1, 7.2, 7.3 and 8.1].
- ▷ Normed Spaces

  Linear operators: boundedness and continuity, finite-dimensional normed spaces, norm compactness of the unit ball, Riesz's lemma for infinite-dimensional normed spaces. [4, Sections 13.1, 13.2, 13.3]
- ▷ Banach Spaces

  Baire category theorem, Open mapping Theorem, Closed graph theorem, Principle of Uniform Boundeness. [4, Sections 13.4, 13.5].
- ▷ Duality for normed spaces

  Linear functionals, Bounded linear functionals, Weak and weak-$*$ topologies, Natural embedding into the second dual, Hahn–Banach theorem, Reflexivity. [4, Sections 14.1, 14.2, 14.3]
- ▷ Locally convex topological vector spaces

  Separation of convex sets, Mazur's theorem. [4, Sections 14.4, 14.5]
- ▷ Compactness of the unit ball

  Helly's theorem, Alaoglu's theorem. [4, Sections 14.3, 15.1]
- ▷ Hilbert spaces

  Riesz representation theorem, Bessel's inequality, Orthonormal bases. [4, Sections 16.2, 16.3]
- ▷ Continuous linear operators on Hilbert spaces.

  Adjoint of a bounded linear operator, Self-adjoint operators, Compact operators, Hilbert–Schmidt theorem. [4, Sections 16.4, 16.5, 16.6]

[4] H. L. Royden and P. M. Fitzpatrick, *Real analysis*, 4th ed., 1988. [QA331.5 .R6 2010].

(f) **Measure Theory and Integration**

- σ-algebras of sets, Measures and measure spaces, G_δ, F_σ and Borel sets in a topological space, Measurable functions, Simple function approximation to a measurable function, The concept of almost everywhere (almost all), Cantor sets and Cantor functions
- Integration on a measure space, Elementary properties of integrals, Convergence theorems (Monotone, Dominated, Fatou's lemma)
- Signed measures, Complex measures, Absolute continuity and singularity for measures, Total variation of a measure, Hahn and Jordan decompositions, Radon Nikodym Theorem and Lebesgue Decomposition Theorem
- Outer measures, Measures derived from outer measures, Extension of a measure on an algebra (semi-algebra) to a measure on a σ-algebra, Caratheodory Extension Theorem
- Product measures, Fubini and Tonneli Theorems, Lebesgue measure and integration on $\mathbb{R}^n$
- General L^p-spaces, Completeness and duality, Riesz representation theorem for the dual of $L^p(X, \mu)$

References:

The topics listed above for this section may be found in Chapters 1-5 and 17-19 of [5]. They are essentially the contents of the course MATH 4260/7260.

[5] H. L. Royden and P. M. Fitzpatrick, *Real analysis*, 4th ed., 1988. [QA331.5 .R6 2010].